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Abstract

This paper models the dynamics of a large set of interacting neurons within the framework of statistical field theory. We use a method initially developed in the context of statistical field theory [47] and later adapted to complex systems in interaction [48][49]. Our model keeps track of individual interacting neurons' dynamics but also preserves some of the features and goals of neural field dynamics, such as indexing a large number of neurons by a space variable. This paper thus bridges the scale of individual interacting neurons and the macro-scale modelling of neural field theory.

1 Introduction

Bridging micro and macro behaviors remains largely problematic for systems with large number of degrees of freedom. When studying neural activity, we either directly start from a macro description of the system, or from a micro description that is then treated numerically.

At the macroscopic scale, mean - or neural - fields, that model large populations of neurons as homogeneous structures and index individual neurons by some spatial coordinates can describe several patterns of brain activity. Following Wilson, Cowan and Amari ([1][2][3][4][5][6][7][8][9]), neural fields dynamics is usually studied in the continuum limit and neural activity is represented by a macroscopic variable, the population-averaged firing rate. The Mean Field approach is an effective theory in which degrees of freedom of some underlying processes are aggregated.

Mean field theory has been extended along various lines and has a wide range of applications.

It allows for travelling wave solutions (see [20][21] and references therein). Stochastic effects in firing rates may be introduced [10][11][12][13][14] to model perturbations and diffusion patterns in the pulse waves dynamics and account for noisy transitions between different mean field regimes (see [15]). Besides, mean fields can be extended to study the impact of neural network topology on spatial configurations of neural activity (see [17], developments in [18], and references therein).

Last but not least, the Mean field approach has been extended using the tools of statistical field theory [19][22][23][24][25][26][27]. Statistical fields stand for the neural activity - or spike counts - at each point of the network. Because it keeps track of covariances between neural activity at different points, the perturbation expansion of the effective action goes beyond mean field approximation. However, since the fields considered represent densities of activity, this extension of mean field theory remains at the collective level rather than deriving from the microscopic features of the network.

Despite the convenience and applications of the mean field formalim, it uses simplifying assumptions to account for the microscopic level, such as delays in interactions or variations of neurons connectivity. Besides, they cannot account for emerging behaviors.

At the microscopic scale, a vast litterature at the intersection of dynamical systems, complex systems and neural networks focuses on neurons dynamics and interactions (see [31][32][33] and references therein).

In strands of litterature such as cognitive neurodynamics or computational neurosciences, neural processes result from the interactions of assemblies of individual neurons. The lower scale allows for a finer

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account of the interrelation between neurons' connectivity and firing rates than the one of neural fields. Usually, no spatial indices are assumed: neurons are not positioned in a spatial structure, and the resolution of the model relies on numerical studies. This approach accounts for neurons' cyclical dynamics, changes in oscillation regimes (for an account, see [31] and references therein) and, more important to us, for the emergence of local connectivity and higher scale phenomena, such as binding problem or polychronization ([36][37][38][39][40][41][42][43][44][45][46]). However, unlike mean fields, these models lack an analytical treatment of collective effect.

The present method bridges the gap between the macro-scale modelling of neural field theory and the assembly of interacting neurons. It is based on a method initially developed in [47] and later adapted to complex systems in interaction [48][49][50][51]. Our model of statistical field theory keeps track of individual interacting neurons dynamics while preserving some features and goals of neural field dynamics. For instance, neurons are indexed by a space variable, to find continuous dynamic equations for the whole system. But, unlike Mean Field Theory and its extensions, our fields do not directly describe any neural activity. As in Statistical Field Theory (see [47]), they are rather abstract complex valued functionals bearing microscopic information to a larger collective scale. Closer to our approach would be ([28]) and ([29]), that use partition functions for the whole system of neurons, or ([30]), that works with an effective action. Yet these approaches use either simplified assumptions at the micro-level, or *a priori* assumptions about the effective action.

Our approach recovers features studied in some extensions of mean field theory. Our results are inherently stochastic: the field describes interactions of neurons subject to dynamics' uncertainty. We recover some traveling waves patterns. Beyond that, our formalism has several advantages. It highlights the influence of some internal variables on the dynamics of firing rates. It may provide a direct approach to phase transitions phenomena, i.e. the impact of collective patterns on individual ones, by studying the effective action of the system. Besides, it allows for a wide range of extensions.

The present field theory results from a two-step process. In a first step, the standard formalism of the dynamic equations of a large set of interacting neurons ([46]) is modified to account for the dynamic nature of neurons connectivity (see [52]). From this, we deduce the firing frequencies' dynamic equations of a large set of neurons.

In a second step, this set of dynamic equations is transformed into a second-quantized Euclidean field theory (see [48][49][50] for the method). This field description includes both collective and individual aspects of the system. The dynamics of the whole system is encompassed both in the action functional for the field and its associated partition function. To understand the role of collective effects in the system, we compute the effective action of the system using standard techniques of field theory. The minimum of this effective action is the vacuum of the theory, i.e. the background field, in which the system evolves. It depends on some internal parameters and external currents and impacts individual neurons dynamics.

We then compute a local approximation for the individual frequencies' dynamic equations that depends on internal and external parameters and on the background field. Depending on the connectivity between neurons, the wave equation may display some non-linear aspects, such as position-dependent coefficients. In the linear approximation, the dynamics reduces to some wave equation, that is either dissipative, stable or explosive. The presence of the background field stabilizes the frequency equation. Some traveling waves are solutions of the system, which shows the importance of the phase of the system for the wave dynamics. Moreover, the field formalism allows to derive frequencies dynamic equations beyond the local approximation. This stresses the importance of the interdependence of the system of frequencies.

The successive derivatives of the effective action with respect to the field yield the correlation functions of the system. These correlation functions compute the joint probabilities of transition for an arbitrary number of neurons and depend directly on the form of the vacuum. These correlation functions for an arbitrary number of points yield an alternative and complementary description of the system frequencies' equations. They compute a joint probability density for the frequencies at different point. These probabilities depend on time, and this linear dependency reflects the ondulatory behaviour of frequencies, and on the background field. Its presence ensures coordination to some extent between frequencies.

The field formalism is then generalized to include several extensions. We show for instance how to include dynamic equations for the connectivity functions. Although these equations are "classical" differential equations, they could also be described by a field formalism. We also show that two types of neurons, inhibitory and excitatory, may be included and their interactions described by the inclusion of two interacting fields in the model.

This paper is organised as follows. Section 2 describes the individual dynamics of neurons in interaction. Section 3 describes the field theoretic formulation of the model and section 4 computes the effective action of the system. In section 5, we derive the minimum of the effective action. In section 6 we find the general form of the frequencies' equation. Section 7 computes the static equilibrium. In section 8, we derive the differential equation for frequencies in the local approximation and show the existence of traveling waves solutions. We also present some extensions of our model and discuss the implication of these extensions. Section 9 derives the frequencies' equation beyond the local approximation. In section 10, we derive a general form for the correlation functions in presence of strong or weak background field and present their interpretation in term of joined probabilities for frequencies at different points. Section 11 is a conclusion

2 Individual dynamics and probability density of the system

Following [48][49][50], we describe a system of a large number of neurons ($N >> 1$). We define their individual equations. Then, we write a probability density for the configurations of the whole system over time.

2.1 Individual dynamics

We follow the description of [46] for coupled quadratic integrate-and-fire (QIF) neurons, but use the additional hypothesis that each neuron is characterized by its position in some spatial range.

Each neuron's potential $X_i(t)$ satisfies the differential equation:

$$\dot{X}_i(t) = \gamma X_i^2(t) + J_i(t) \tag{1}$$

for $X_i(t) < X_p$, where X_p denotes the potential level of a spike. When $X = X_p$, the potential is reset to its resting value $X_i(t) = X_r < X_p$. For the sake of simplicity, following ([46]) we have chosen the squared form $\gamma X_i^2(t)$ in (1). However any form $f(X_i(t))$ could be used. The current of signals reaching cell i at time t is written $J_i(t)$.

Our purpose is to find the system dynamics in terms of the spikes' frequencies. First, we consider the time for the n-th spike of cell i, $\theta_n^{(i)}$. This is written as a function of n, $\theta^{(i)}(n)$. Then, a continuous approximation $n \to t$ allows to write the spike time variable as $\theta^{(i)}(t)$. We thus have replaced:

$$\theta_n^{(i)} \to \theta^{(i)}(n) \to \theta^{(i)}(t)$$

The continuous approximation could be removed, but is convenient and simplifies the notations and computations. We assume now that the timespans between two spikes are relatively small. The time between two spikes for cell i is obtained by writing (1) as:

$$\frac{dX_i(t)}{dt} = \gamma X_i^2(t) + J_i(t)$$

and by inverting this relation to write:

$$dt = \frac{dX_i}{\gamma X_i^2 + J^{(i)}\left(\theta^{(i)}(n-1)\right)}$$

Integrating the potential between two spikes thus yields:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \simeq \int_{X_r}^{X_p} \frac{dX}{\gamma X^2 + J^{(i)}\left(\theta^{(i)}(n-1)\right)}$$

Replacing $J^{(i)}\left(\theta^{(i)}(n-1)\right)$ by its average value during the small time period $\theta^{(i)}(n) - \theta^{(i)}(n-1)$, we can consider $J^{(i)}\left(\theta^{(i)}(n-1)\right)$ as constant in first approximation, so that:

$$\theta^{(i)}(n)-\theta^{(i)}(n-1) \simeq \frac{\left[\arctan\left(\sqrt{\frac{\gamma}{J^{(i)}\left(\theta^{(i)}(n-1)\right)}}X\right)\right]_{X_r}^{X_p}}{\sqrt{\gamma J^{(i)}\left(\theta^{(i)}(n-1)\right)}}$$

$$= \frac{\left[\arctan\left(\frac{1}{X}\sqrt{\frac{J^{(i)}\left(\theta^{(i)}(n-1)\right)}{\gamma}}\right)\right]_{X_p}^{X_r}}{\sqrt{\gamma J^{(i)}\left(\theta^{(i)}(n-1)\right)}} = \frac{\arctan\left(\frac{\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{\frac{J^{(i)}\left(\theta^{(i)}(n-1)\right)}{\gamma}}}{1+\frac{J^{(n)}\left(\theta^{(n-1)}\right)}{\gamma X_r X_p}}\right)}{\sqrt{\gamma J^{(i)}\left(\theta^{(i)}(n-1)\right)}}$$

For γ normalized to 1 and $\frac{J^{(n)}\left(\theta^{(n-1)}\right)}{X_r X_p} << 1$, this is:

$$\theta^{(i)}(n)-\theta^{(i)}(n-1) \equiv G\left(\theta^{(i)}(n-1)\right) = \frac{\arctan\left(\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}\right)}{\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}} \tag{2}$$

The frequency or firing rate at t, $\omega_i(t)$, is defined by the inverse time span (2) between two spikes:

$$\omega_i(t) = \frac{1}{G\left(\theta^{(i)}(n-1)\right)}$$

$$\equiv F\left(\theta^{(i)}(n-1)\right) = \frac{\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}}{\arctan\left(\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}\right)}$$

Since we consider small time intervals between two spikes, we can write:

$$\theta^{(i)}(n)-\theta^{(i)}(n-1) \simeq \frac{d}{dt}\theta^{(i)}(t)-\omega_i^{-1}(t)=\varepsilon_i(t) \tag{3}$$

where the white noise perturbation $\varepsilon_i(t)$ for each period was added to account for any internal uncertainty in the time span $\theta^{(i)}(n)-\theta^{(i)}(n-1)$. This white noise is independent from the instantaneous inverse frequency $\omega_i^{-1}(t)$. We assume these $\varepsilon_i(t)$ to have variance σ^2, so that equation (3) writes:

$$\frac{d}{dt}\theta^{(i)}(t)-G\left(\theta^{(i)}(t), J^{(i)}\left(\theta^{(i)}(t)\right)\right)=\varepsilon_i(t) \tag{4}$$

The $\omega_i(t)$ are computed by considering the overall current which, using the discrete time notation, is given by:

$$\hat{J}^{(i)}((n-1)) = J^{(i)}((n-1))+\frac{\kappa}{N}\sum_{j,m}\frac{\omega_j(m)}{\omega_i(n-1)}\delta\left(\theta^{(i)}(n-1)-\theta^{(j)}(m)-\frac{|Z_i-Z_j|}{c}\right)T_{ij}((n-1,Z_i),(m,Z_j)) \tag{5}$$

The quantity $J^{(i)}((n-1))$ denotes an external current. The term inside the sum is the average current sent to i by neuron j during the short time span $\theta^{(i)}(n)-\theta^{(i)}(n-1)$. The function $T_{ij}((n-1,Z_i),(m,Z_j))$ is the transfer function between cells j and i. It measures the level of connectivity between i and j. We assume that:

$$T_{ij}((n-1,Z_i),(m,Z_j)) = T((n-1,Z_i),(m,Z_j))$$

The transfer function of Z_j on Z_i only depends on positions and times. It models the transfer function as an average transfer between local zones of the thread. This transfer function is typically considered as gaussian or decreasing exponentially with the distance between neurons, so that the closer the cells, the more connected they are.

We can justify the other terms arising in (5): given the distance $|Z_i - Z_j|$ between the two cells and the signals' velocity c, signals arrive with a delay $\frac{|Z_i - Z_j|}{c}$. The spike emitted by cell j at time $\theta^{(j)}(m)$ has thus to satisfy:

$$\theta^{(i)}(n-1) < \theta^{(j)}(m) + \frac{|Z_i - Z_j|}{c} < \theta^{(i)}(n)$$

to reach cell i during the timespan $\left[\theta^{(i)}(n-1), \theta^{(i)}(n)\right]$. This relation must be represented by a step function in the current formula. However given our approximations, this can be replaced by:

$$\delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c}\right)$$

as in (5). However, this Dirac function must be weighted by the number of spikes emitted during the rise of the potential. This number is the ratio $\frac{\omega_j(m)}{\omega_i(n-1)}$ that counts the number of spikes emitted by neuron j towards neuron i between the spikes $n-1$ and n of neuron i. Again, this is valid for relatively small timespans between two spikes. For larger timespans, the frequencies should be replaced by their average over this period of time.

The sum over m and i is the overall contribution to the current from any possible spike of the thread, provided it arrives at i during the interval $\theta^{(i)}(n) - \theta^{(i)}(n-1)$ considered. Note that the current (5) is partly an endogenous variable. It depends on signals external to i, but depends also on i through $\omega_i(n-1)$. This is a consequence of the intrication between the system's elements.

In the sequel, we will work in the continuous approximation, so that formula (5) is replaced by:

$$\hat{J}^{(i)}(t) = J^{(i)}(t) + \frac{\kappa}{N}\int \sum_j \frac{\omega_j(s)}{\omega_i(t)}\delta\left(\theta^{(i)}(t) - \theta^{(j)}(s) - \frac{|Z_i - Z_j|}{c}\right) T_{ij}\left((t, Z_i), (s, Z_j)\right) ds \tag{6}$$

Formula (6) shows that the dynamic equation (3) has to be coupled with the frequency equation:

$$\begin{aligned}\omega_i(t) &= G\left(\theta^{(i)}(t), \hat{J}\left(\theta^{(i)}(t)\right)\right) + \upsilon_i(t) \\ &= \frac{\sqrt{\hat{J}^{(i)}(t)}}{\arctan\left(\left(\frac{1}{X_r} - \frac{1}{X_p}\right)\sqrt{\hat{J}^{(i)}(t)}\right)} + \upsilon_i(t)\end{aligned} \tag{7}$$

and $J^{(i)}(t)$ is defined by (6). A white noise $\upsilon_i(t)$ accounts for the possible deviations from this relation, due to some internal or external causes for each cell. We assume that the variances of $\upsilon_i(t)$ are constant, and equal to η^2, such that $\eta^2 << \sigma^2$.

2.2 Probability density for the system

Due to the stochastic nature of equations (4) and (7), the dynamics of a single neuron can be described by the probability density $P\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right)$ for a path $\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right)$ which is given by, up to a normalization factor:

$$P\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right) = \exp(-A_i) \tag{8}$$

where:

$$A_i = \frac{1}{\sigma^2}\int\left(\frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t)\right)^2 dt + \int \frac{\left(\omega_i^{-1}(t) - G\left(\theta^{(i)}(t), \hat{J}\left(\theta^{(i)}(t)\right)\right)\right)^2}{\eta^2} dt \tag{9}$$

(see [48] and [49]). The integral is taken over a time period that depends on the time scale of the interactions. Actually, the minimization of (9) imposes both (3) and (7), so that the probability density is, as expected, centered around these two conditions, i.e. (3) and (7) are satisfied in mean. A probability density for the whole system is obtained by summing S_i over all agents. We thus define:

$$P\left(\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right)_{i=1\ldots N}\right) = \exp(-A) \tag{10}$$

with:

$$A=\sum_i A_i=\sum_i \frac{1}{\sigma^2}\int\left(\frac{d}{dt}\theta^{(i)}(t)-\omega_i^{-1}(t)\right)^2 dt+\int\frac{\left(\omega_i^{-1}(t)-G\left(\theta^{(i)}(t),\hat{J}\left(\theta^{(i)}(t)\right)\right)\right)^2}{\eta^2}dt \tag{11}$$

3 Field theoretic description of the system

3.1 translation of Equation (11) in terms of field theory

We have shown in [48][49][50] that the probabilistic description of the system (10) is equivalent to a statistical field formalism. In such a formalism, the system is collectively described by a field that is an element of the Hilbert space of complex functions. The arguments of these functions are the same as those describing an individual neuron. A shortcut of the translation of systems similar to (11) in terms of field, is given in [51] . The next paragraph gives an account of this method.

3.1.1 Principle

General form of the statistical weight In general, we assume a system in which individual agents (here cells) are described by vectors $\mathbf{X}_i(t)$ of arbitrary dimension, and such that the exponent of the statistical weight of the system has the form:

$$\sum_i\int\left(\frac{d\mathbf{X}_i^{(\alpha)}(t)}{dt}-\int\sum_{j,k,l...}f^{(\alpha)}\left(\mathbf{X}_i(t),\mathbf{X}_j(t_j),\mathbf{X}_k(t_k)...\right)dt_idt_jdt_k...\right)^2 dt \tag{12}$$
$$+\sum_i\int\sum_{j,k,l...}g\left(\mathbf{X}_i(t_i),\mathbf{X}_j(t_j),\mathbf{X}_k(t_k)...\right)dt_idt_jdt_k...$$

for some functions $f^{(\alpha)}$ and g. The introduction of index α represents the dynamics of the α-th coordinate of a variable $\mathbf{X}_i(t)$ as a function of the other agents. If we replace $\hat{J}\left(\theta^{(i)}(t)\right)$ in (11) by its expression (6), we can check that (11) has the form (12). This point is detailed below.

Translation in terms of fields The translation itself can be divided into two relatively simple processes, but varies slightly depending on the type of terms that appear in the various minimization functions.

Term without temporal derivative The terms in (12) that include indexed variables but no temporal derivative terms are the easiest to translate. They are of the form:

$$\sum_i\int\sum_{j,k,l...}g\left(\mathbf{X}_i(t_i),\mathbf{X}_j(t_j),\mathbf{X}_k(t_k)...\right)dt_idt_jdt_k...$$

These terms describe the whole set of interactions between agents characterized by their variables $\mathbf{X}_i(t),\mathbf{X}_j(t),\mathbf{X}_k(t)$...

In the field translation, agents are described by a field $\Psi(\mathbf{X})$ where $\mathbf{X}$ is a vector of the same dimension as the $\mathbf{X}_i$.

In a first step, the variables indexed i such as $\mathbf{X}_i(t)$ are replaced by variables $\mathbf{X}$ in the expression of g. The variables indexed j,k,l,m..., such as $\mathbf{X}_j(t)$, $\mathbf{X}_k(t)$... are replaced by $\mathbf{X}',\mathbf{X}''$, and so on for all the indices in the function. This yields the expression:

$$\sum_i\sum_{j,k,l,m...}g\left(\mathbf{X},\mathbf{X}',\mathbf{X}''...\right)$$

In a second step, each sum is replaced by a weighted integration symbol:

$$\sum_i\rightarrow\int\left|\Psi(\mathbf{X})\right|^2d\mathbf{X},\ \sum_j\rightarrow\int\left|\Psi(\mathbf{X}')\right|^2d\mathbf{X}',\ \sum_k\rightarrow\int\left|\Psi(\mathbf{X}'')\right|^2d\mathbf{X}''$$

which leads to the translation:

$$\begin{aligned}&\sum_i \int \sum_{j,k,l...} g\left(\mathbf{X}_i\left(t_i\right), \mathbf{X}_j\left(t_j\right), \mathbf{X}_k\left(t_k\right)...\right) dt_i dt_j dt_k... \\ \rightarrow &\int g\left(\mathbf{X}, \mathbf{X}', \mathbf{X}''...\right) \left|\Psi\left(\mathbf{X}\right)\right|^2 \left|\Psi\left(\mathbf{X}'\right)\right|^2 \left|\Psi\left(\mathbf{X}''\right)\right|^2 d\mathbf{X} d\mathbf{X}' d\mathbf{X}''...\end{aligned} \tag{13}$$

Note that this formula can be generalized if we compose the previous formula by any function G. The translation of an expression of the form:

$$\sum_i \int G\left(\int \sum_{j,k,l...} g\left(\mathbf{X}_i\left(t_i\right), \mathbf{X}_j\left(t_j\right), \mathbf{X}_k\left(t_k\right)...\right) dt_j dt_k...\right) dt_i \tag{14}$$

is obtained by expanding (14) in powers of $\int \sum_{j,k,l...} g\left(\mathbf{X}_i\left(t_i\right), \mathbf{X}_j\left(t_j\right), \mathbf{X}_k\left(t_k\right)...\right) dt_j dt_k...$ and using (13). We find:

$$\begin{aligned}&\sum_i \int G\left(\int \sum_{j,k,l...} g\left(\mathbf{X}_i\left(t_i\right), \mathbf{X}_j\left(t_j\right), \mathbf{X}_k\left(t_k\right)...\right) dt_j dt_k...\right) dt_i \\ \rightarrow &\int \left|\Psi\left(\mathbf{X}\right)\right|^2 G\left(\int g\left(\mathbf{X}, \mathbf{X}', \mathbf{X}''...\right) \left|\Psi\left(\mathbf{X}'\right)\right|^2 \left|\Psi\left(\mathbf{X}''\right)\right|^2 d\mathbf{X}' d\mathbf{X}''...\right) d\mathbf{X}\end{aligned} \tag{15}$$

Term with temporal derivative The terms in (12) that imply a variable temporal derivative are of the form:

$$\sum_i \int \left(\frac{d\mathbf{X}_i^{(\alpha)}\left(t\right)}{dt} - \int \sum_{j,k,l...} f^{(\alpha)}\left(\mathbf{X}_i\left(t\right), \mathbf{X}_j\left(t_j\right), \mathbf{X}_k\left(t_k\right)...\right) dt_i dt_j dt_k\right)^2 dt \tag{16}$$

The method of translation is similar to the above, but the time derivative adds an additional operation.

In a first step, we translate the terms without derivative inside the parenthesis:

$$\int \sum_{j,k,l...} f^{(\alpha)}\left(\mathbf{X}_i\left(t\right), \mathbf{X}_j\left(t_j\right), \mathbf{X}_k\left(t_k\right)...\right) dt_i dt_j dt_k \tag{17}$$

The translation of this type of term has already been presented in the previous paragraph. Note however that, in (17), there is no sum over i, so that the translation includes neither the integral over X, nor the factor $\left|\Psi\left(\mathbf{X}\right)\right|^2$.

The translation of (17) is therefore, as before:

$$\int f^{(\alpha)}\left(\mathbf{X}, \mathbf{X}', \mathbf{X}''...\right) \left|\Psi\left(\mathbf{X}'\right)\right|^2 \left|\Psi\left(\mathbf{X}''\right)\right|^2 d\mathbf{X}' d\mathbf{X}'' \tag{18}$$

A free variable $\mathbf{X}$ remains, which will be integrated later, when we account for the external sum $\sum_i$. We will call $\Lambda(\mathbf{X})$ the expression obtained in (18):

$$\Lambda(\mathbf{X}) = \int f^{(\alpha)}\left(\mathbf{X}, \mathbf{X}', \mathbf{X}''...\right) \left|\Psi\left(\mathbf{X}'\right)\right|^2 \left|\Psi\left(\mathbf{X}''\right)\right|^2 d\mathbf{X}' d\mathbf{X}'' \tag{19}$$

In a second step, we account for the derivative in time by using field gradients. To do so, and as a rule, we replace :

$$\sum_i \left(\frac{d\mathbf{X}_i^{(\alpha)}\left(t\right)}{dt} - \int \sum_{j,k,l...} f^{(\alpha)}\left(\mathbf{X}_i\left(t\right), \mathbf{X}_j\left(t_j\right), \mathbf{X}_k\left(t_k\right)...\right) dt_i dt_j dt_k\right)^2 \tag{20}$$

by:

$$\int \Psi^{\dagger}\left(\mathbf{X}\right) \left(-\nabla_{\mathbf{X}^{(\alpha)}} \left(\frac{\sigma^2_{\mathbf{X}^{(\alpha)}}}{2} \nabla_{\mathbf{X}^{(\alpha)}} + \Lambda(\mathbf{X})\right)\right) \Psi\left(\mathbf{X}\right) d\mathbf{X} \tag{21}$$

The variance $\sigma^2_{\mathbf{X}^{(\alpha)}}$ reflects the probabilistic nature of the model which is hidden behind the field formalism. This variance represents the characteristic level of uncertainty of the system's dynamics. It is a parameter of the model. Note also that in (21), the integral over $\mathbf{X}$ reappears at the end, along with the square of the field $|\Psi(\mathbf{X})|^2$. This square is split into two terms, $\Psi^\dagger(\mathbf{X})$ and $\Psi(\mathbf{X})$, with a gradient operator inserted in between.

3.1.2 Translation of (11)

In our context, the field depends on the three variables (θ, Z, ω), and writes $\Psi(\theta, Z, \omega)$. The field dynamics is described by an action functional for the field and its associated partition function. This partition function reflects both collective and individual aspects of the system, and allows to recover correlation functions for an arbitrary number of neurons.

The field theoretic version of (9) is obtained using (11): a correspondence detailed in [48][49]) yields an action $S(\Psi)$ for a field $\Psi(\theta, Z, \omega)$ and a statistical weight $\exp(-(S(\Psi)))$ for each configuration $\Psi(\theta, Z, \omega)$ of this field. The functional $S(\Psi)$ is decomposed in two parts corresponding to the two contributions in (11).

The first term of (11):

$$\frac{1}{\sigma^2}\int\left(\frac{d}{dt}\theta^{(i)}(t)-\omega_i^{-1}(t)\right)^2 dt \tag{22}$$

is a term with temporal derivative. Its form is simple since the function $f^{(\alpha)}$ in (20) depends only on the variable $\mathbf{X}_i(t)=\left(\theta^{(i)}(t), \omega_i^{-1}(t), Z_i\right)$. Actually $f^{(\theta)}(\mathbf{X}_i(t))=\omega_i^{-1}(t)$. Using (21), the term (22) is thus replaced by the corresponding quadratic functional in field theory :

$$-\frac{1}{2}\Psi^\dagger(\theta, Z, \omega)\nabla\left(\frac{\sigma^2}{2}\nabla-\omega^{-1}\right)\Psi(\theta, Z, \omega) \tag{23}$$

where σ^2 is the variance of the errors ε_i.

The field functional that corresponds to the second term of (9):

$$V=\int\frac{\left(\omega_i^{-1}(t)-G\left(\theta^{(i)}(t), \hat{J}\left(\theta^{(i)}(t)\right)\right)\right)^2}{\eta^2}dt$$

is obtained by expanding the formula (6) for the current induced by all j:

$$\begin{aligned} V &= \frac{1}{2\eta^2}\int dt\sum_i\left(\omega_i^{-1}(t)\right. \\ &\quad\left.-G\left(J\left(\theta^{(i)}(t), Z_i\right)+\frac{\kappa}{N}\int ds\sum_j\frac{\omega_j(s)\,T_{ij}\left((t, Z_i), s, Z_j\right)}{\omega_i(t)}\delta\left(\theta^{(i)}(t)-\theta^{(j)}(s)-\frac{|Z_i-Z_j|}{c}\right)\right)\right)^2 \end{aligned} \tag{24}$$

with $\eta << 1$, which is the constraint (7) imposed stochastically. Its corresponding potential in field theory is obtained straightforwardly by using the translation (15):

$$\frac{1}{2\eta^2}\int|\Psi(\theta, Z, \omega)|^2\left(\omega^{-1}-G\left(J(\theta, Z)+\int\frac{\kappa}{N}\frac{\omega_1 T\left(Z, \theta, Z_1, \theta-\frac{|Z-Z_1|}{c}\right)}{\omega}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c}, Z_1, \omega_1\right)\right|^2 dZ_1 d\omega_1\right)\right)^2 \tag{25}$$

To simplify, we will write in the sequel:

$$T\left(Z, \theta, Z_1, \theta-\frac{|Z-Z_1|}{c}\right)\equiv T(Z, \theta, Z_1)$$

The field action is then the sum of (23) and (25):

$$S = -\frac{1}{2}\Psi^{\dagger}(\theta,Z,\omega)\nabla\left(\frac{\sigma_{\theta}^{2}}{2}\nabla-\omega^{-1}\right)\Psi(\theta,Z,\omega) \tag{26}$$
$$+\frac{1}{2\eta^{2}}\int\left|\Psi(\theta,Z,\omega)\right|^{2}\left(\omega^{-1}-G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega_{1}}{\omega}\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\omega_{1}\right)\right|^{2}T(Z,\theta,Z_{1})\,dZ_{1}d\omega_{1}\right)\right)^{2}$$

3.2 Projection on dependent frequency states:

Using the fact that $\eta^2 << 1$, and noting that in this case, field configurations $\Psi(\theta,Z,\omega)$ such that:

$$\omega^{-1}-G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega_{1}}{\omega}\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\omega_{1}\right)\right|^{2}T(Z,\theta,Z_{1})\,dZ_{1}d\omega_{1}\right)\neq 0$$

have negligible statistical weight, we can simplify (26) and restrict the fields to those of the form:

$$\Psi(\theta,Z)\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\right) \tag{27}$$

where $\omega^{-1}(J,\theta,Z,\Psi)$ satisfies:

$$\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right) = G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega_{1}T\left(Z,\theta,Z_{1},\theta-\frac{|Z-Z_{1}|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)}\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\omega_{1}\right)\right|^{2}dZ_{1}d\omega_{1}\right)$$
$$= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega_{1}T\left(Z,\theta,Z_{1},\theta-\frac{|Z-Z_{1}|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)}\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}\right.$$
$$\left.\times\,\delta\left(\omega_{1}^{-1}-\omega^{-1}\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1},|\Psi|^{2}\right)\right)dZ_{1}d\omega_{1}\right)$$

The last equation simplifies to yield:

$$\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)=G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\Psi\right)T\left(Z,\theta,Z_{1},\theta-\frac{|Z-Z_{1}|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)}\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}dZ_{1}\right) \tag{28}$$

The configurations $\Psi(\theta,Z,\omega)$ that minimize the potential (25) can now be considered: the field $\Psi(\theta,Z,\omega)$ is projected on the subspace (27) of functions of two variables, and we can therefore replace in (25):

$$\omega^{-1}\rightarrow\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)$$

The "classical" effective action becomes (see appendix 0):

$$-\frac{1}{2}\Psi^{\dagger}(\theta,Z)\left(\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\right)\right)\Psi(\theta,Z) \tag{29}$$

with $\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)$ given by equation (28).

The form of the transfer function $T(Z,\theta,Z_1)$ can ultimately be refined. Using a simplified version of [52], appendix 6 shows that, at the individual level and in first approximation, the transfer functions are modelled by a product between a spatial factor $T(Z,Z_1)$ and a function W of the frequencies $\omega\equiv\omega\left(J,\theta,Z,|\Psi|^{2}\right)$, and $\omega_1\equiv\omega\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1},|\Psi|^{2}\right)$. The function W is increasing in ω and decreasing in ω_1. Without loss of generality, we will consider W as an increasing function of $\left(\frac{\omega}{\omega_{1}}\right)$, so that:

$$T(Z,\theta,Z_{1})=T(Z,Z_{1})\,W\left(\frac{\omega}{\omega_{1}}\right) \tag{30}$$

3.3 Inclusion of collective stabilization potential

To stabilize the number of active connexions, we ultimately modify (29) by including collective terms. These terms correspond to some overall regulatory processes that are not accounted for in the model in its actual form. When there is no "competition" between inhibitory and excitatory mechanisms, such a potential models the ability for a system to return to some minimal equilibrium activity.

To do so, we add to the action functional (29) a potential $V(\Psi)$ that maintains and activates new connections. The system's action thus becomes:

$$S = -\frac{1}{2}\Psi^{\dagger}(\theta, Z)\left(\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta} - \omega^{-1}\left(J, \theta, Z, |\Psi|^{2}\right)\right)\right)\Psi(\theta, Z) + V(\Psi) \tag{31}$$

We choose the following form for $V(\Psi)$:

$$V(\Psi) = \int |\Psi(\theta, Z)|^{2} U_{0}\left(\int \left|\Psi\left(\theta - \frac{|Z - Z'|}{c}, Z'\right)\right|^{2}\right) \tag{32}$$

where U_0 is a U shaped potential, with $U_0(0) = 0$ so that U_0 has a minimum for some positive value of $\left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^{2}$. At this minimum, the potential U_0 is negative.

Note that, expression (32) models the interactions between activity at time θ of cells located at Z, and activity at time $\theta - \frac{|Z-Z'|}{c}$ of those located at any point Z'. The delay $\frac{|Z-Z'|}{c}$ in time is induced by travelling time of signals between Z and Z'.

The potential (32) can be written as a series expansion. We choose the following decomposition:

$$V(\Psi) = -\zeta_{1}\int\left(|\Psi(\theta, Z)|^{2}\left|\Psi\left(\theta - \frac{|Z - Z'|}{c}, Z'\right)\right|^{2}\right) + \sum_{n=2}^{\infty}\zeta_{n}\int|\Psi(\theta, Z)|^{2}\left(\prod_{i=1}^{n-1}\left|\Psi\left(\theta - \frac{|Z - Z_{i}|}{c}, Z_{i}\right)\right|^{2}\right) \tag{33}$$

The first term in (33):

$$-\zeta_{1}\int\left(|\Psi(\theta, Z)|^{2}\left|\Psi\left(\theta - \frac{|Z - Z'|}{c}, Z'\right)\right|^{2}\right)$$

with $\zeta_1 > 0$, accounts for a minimal number of connections that are permanently maintained. The magnitude of this factor depends on the external activity J.

The second term in (33) models a global limitation mechanism. It increases with the overall number of connections and currents, so that we assume for $n \geqslant 2$:

$$\sum_{k=2}^{n}\zeta_{k}\left\langle\prod_{i=1}^{k-1}\left|\Psi\left(\theta - \frac{|Z - Z_{i}|}{c}, Z_{i}\right)\right|^{2}\right\rangle > 0$$

where the bracket $\langle\rangle$ denotes the expectation value of the product of fields.

The coefficients ζ_n can be set to 0 for $n > N$, where N is an arbitrary threshold. The term proportional to $-\zeta_1$ and the contribution for $n = 2$ can be gathered to rewrite the collective potential as:

$$V(\Psi) = \sum_{n=2}^{\infty}\zeta^{(n)}\int|\Psi(\theta, Z)|^{2}\left(\prod_{i=1}^{n-1}\left|\Psi\left(\theta - \frac{|Z - Z_{i}|}{c}, Z_{i}\right)\right|^{2}\right) \tag{34}$$

where $\zeta^{(n)} = \zeta_n$ for $n > 2$, and $\zeta^{(2)} = \zeta_2 - \zeta_1$. We assume that $\zeta^{(2)} < 0$, so that a nontrivial minimal collective state exists. The classical action is thus:

$$\begin{aligned} & -\frac{1}{2}\Psi^{\dagger}(\theta, Z)\left(\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta} - \omega^{-1}\left(J, \theta, Z, |\Psi|^{2}\right)\right)\right)\Psi(\theta, Z) + V(\Psi) \\ \equiv & -\frac{1}{2}\Psi^{\dagger}(\theta, Z)\left(\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta} - \omega^{-1}\left(J, \theta, Z, |\Psi|^{2}\right)\right)\right)\Psi(\theta, Z) + \sum_{n=2}^{\infty}\zeta^{(n)}\int|\Psi(\theta, Z)|^{2}\left(\prod_{i=1}^{n-1}\left|\Psi\left(\theta - \frac{|Z - Z_{i}|}{c}, Z_{i}\right)\right|^{2}\right) \end{aligned} \tag{35}$$

Since we are merely interested in the relative magnitudes of the coefficients σ_θ^2 and quantities such as ω^{-1}, we can impose a constraint on the coefficients $\zeta^{(n)}$. As a relative benchmark, we choose:

$$\begin{aligned}&\sum_{n=2}^{\infty}\zeta^{(n)}\int\left|\Psi\left(\theta,Z\right)\right|^{2}\left(\prod_{i=1}^{n-1}\left\langle\left|\Psi\left(\theta-\frac{\left|Z-Z_{i}\right|}{c},Z_{i}\right)\right|^{2}\right\rangle\right)\\ =&\int\left|\Psi\left(\theta,Z\right)\right|^{2}U_{0}\left(\int\left\langle\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^{2}\right\rangle\right)\simeq 1\end{aligned}$$

Several extensions of the formalism will be considered. The details are left for further research.

3.4 Including excitatory vs inhibitory interactions

The previous formalism can be extended to include inhibitory currents. To do so, two different types of cells, each defined by a different field, are introduced. We write $\Psi_1(\theta,Z,\omega)$ and $\Psi_2\left(\theta,\tilde{Z},\tilde{\omega}\right)$ for excitatory and inhibitory neurons respectively. A straightforward generalization to an arbitrary number of types of cells is presented at the end of the section.

3.4.1 Asymetric interaction between two types of cells

We consider two types of cells. One set is composed of interacting cells, as described by the previous formalism, while the other acts as inhibitor or regulator on the first set. The influence of each type of cell on the other translates through the actions of the induced currents. Assuming identical transfer functions for both types of fields, the corresponding potential terms for the frequencies are, for the first type of cells:

$$\begin{aligned}&\frac{1}{2\eta^{2}}\int\left|\Psi_{1}\left(\theta,Z,\omega\right)\right|^{2}\left(\omega^{-1}-G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega_{1}}{\omega}\left|\Psi_{1}\left(\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1},\omega_{1}\right)\right|^{2}T\left(Z,\theta,Z_{1}\right)dZ_{1}d\omega_{1}\right.\right.\\&\left.\left.-\int\frac{\kappa}{N}\frac{\tilde{\omega}_{1}}{\omega}\left|\Psi_{2}\left(\theta-\frac{\left|Z-\tilde{Z}_{1}\right|}{c},\tilde{Z}_{1},\tilde{\omega}_{1}\right)\right|^{2}T\left(Z,\theta,\tilde{Z}_{1}\right)d\tilde{Z}_{1}d\tilde{\omega}_{1}\right)\right)^{2}\end{aligned}\tag{36}$$

and, for the second type:

$$\frac{1}{2\eta^{2}}\int\left|\Psi_{2}\left(\theta,\tilde{Z},\tilde{\omega}\right)\right|^{2}\left(\tilde{\omega}^{-1}-G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega_{1}}{\tilde{\omega}}\left|\Psi_{1}\left(\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1},\omega_{1}\right)\right|^{2}T\left(Z,\theta,Z_{1}\right)dZ_{1}d\omega_{1}\right)\right)^{2}\tag{37}$$

This models the fact that the second system merely inhibits the first one.

As in section 3.2, we project the fields on the frequency-dependent states defined by (36) and (37). The resulting action for the system is:

$$\begin{aligned}S&=&-\frac{1}{2}\Psi_{1}^{\dagger}\left(\theta,Z\right)\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J,\theta,Z,\Psi_{1},\Psi_{2}\right)\right)\Psi_{1}\left(\theta,Z\right)\\&&-\frac{1}{2}\Psi_{2}^{\dagger}\left(\theta,\tilde{Z}\right)\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\tilde{\omega}^{-1}\left(J,\theta,Z,\Psi_{1},\Psi_{2}\right)\right)\Psi_{2}\left(\theta,\tilde{Z}\right)\end{aligned}\tag{38}$$

where the frequencies satisfy:

$$\begin{aligned}\omega^{-1}(J,\theta,Z,\Psi_1,\Psi_2) &= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)}{\omega}\left|\Psi_1\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 T(Z,\theta,Z_1)\,dZ_1\right.\\&\left.-\int\frac{\kappa}{N}\frac{\tilde{\omega}\left(J,\theta-\frac{|Z-Z_1|}{c},\tilde{Z}_1,\Psi\right)}{\omega}\left|\Psi_2\left(\theta-\frac{\left|Z-\tilde{Z}_1\right|}{c},\tilde{Z}_1\right)\right|^2 T\left(Z,\theta,\tilde{Z}_1\right)d\tilde{Z}_1\right) \qquad (39)\\ \tilde{\omega}^{-1}(J,\theta,Z,\Psi_1,\Psi_2) &= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\tilde{\omega}}\right.\\&\left.\times\left|\Psi_2\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1\right) \qquad (40)\end{aligned}$$

Ultimately, a collective potential can be added, as in section 3.3:

$$\sum_{n=2}^{\infty}\zeta^{(n)}\int|\Psi(\theta,Z)|^2\left(\prod_{i=1}^{n-1}\left|\Psi\left(\theta-\frac{|Z-Z_i|}{c},Z_i\right)\right|^2\right) \qquad (41)$$

where we define:

$$|\Psi(\theta,Z)|^2=|\Psi_1(\theta,Z)|^2+|\Psi_2(\theta,Z)|^2$$

Potential (41) models an equilibrium that results from both excitatory and inhibitory activities.

3.4.2 n interacting fields

The previous 2-fields description may be generalized to describe n interacting types of cells, with arbitrary interactions. Each type of cells is caracterized by its frequency $i=1,...,n$, and interacts either positively or negatively with each other. Each type is defined by a field Ψ_i and frequencies $\omega_i(\theta,Z)$. The general version of (38), that includes (41), becomes:

$$\begin{aligned}S &= -\frac{1}{2}\sum_i\Psi_i^{\dagger}(\theta,Z)\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega_i^{-1}(J,\theta,Z,\Psi_1,\Psi_2)\right)\Psi_i(\theta,Z) \qquad (42)\\&+\sum_i\sum_{n=2}^{\infty}\zeta_i^{(n)}\int|\Psi_i(\theta,Z)|^2\left(\prod_{i=1}^{n-1}\left|\Psi_i\left(\theta-\frac{|Z-Z_i|}{c},Z_i\right)\right|^2\right)\end{aligned}$$

and equations for frequencies are defined by:

$$\begin{aligned}\omega_i(\theta,Z) &= F_i\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega_i(\theta,Z)}G^{ij}\right. \qquad (43)\\&\left.\times W\left(\frac{\omega_i(\theta,Z)}{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\bar{\mathcal{G}}_{0j}(0,Z_1)+\left|\Psi_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

The $n\times n$ matrix G has coefficients in the interval $[-1,1]$. In the sequel, the sum over index j is implicit. For instance, if $n=2$, the matrix g:

$$G=\begin{pmatrix}1 & -g\\ -g & 0\end{pmatrix}$$

represents inhibitory interactions between the two populations defined in (36) and (37).

In the sequel, the computations will focus on the one-field basic model. The implications for several fields model will be discussed at the end of the paper.

4 Effective action

4.1 Principle

Appendices 1, 2 and 3 present the computation of the effective action associated to the action functional (35). To do so, appendix 1 computes the two-points Green functions of the system from which the lowest order expansion in power of field of the effective action is derived, while appendix 2 considers the whole series of graphs. Appendix 3 then provides a compact expression for the effective action.

4.2 Effective action at the tree order

The effective action at the lowest order in powers of fields is computed through the two-points Green function, that are computed by a graphs expansion with free propagator.

4.2.1 Propagator for the free action

Appendix 1.1.2 and 1.1.3 compute the two-points Green functions associated to (35) by a graph expansion using the propagator associated to the "free action":

$$-\Psi^{\dagger}(\theta, Z)\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta}-\omega^{-1}(J, \theta, Z, 0)\right)\Psi(\theta, Z) \tag{44}$$

where $\omega^{-1}(J, \theta, Z, 0)$ is the inverse frequency given by (28) for $\Psi \equiv 0$, i.e. $\omega^{-1}(J, \theta, Z, \Psi) = G(J(\theta, Z))$. Action (35) thus decomposes as:

$$-\frac{1}{2}\Psi^{\dagger}(\theta, Z)\left(\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta}-\omega^{-1}(J, \theta, Z, 0)\right)\right)+\frac{1}{2}\Psi^{\dagger}(\theta, Z)\nabla_{\theta}\omega^{-1}\left(J, \theta, Z, |\Psi|^{2}\right)\Psi(\theta, Z)+V(\Psi) \tag{45}$$

and the two last terms in the previous expression are considered perturbatively in the computation of the graphs.

We find in appendix 1.1.1 that for an external current decomposed in a static and dynamic part $\bar{J}(Z)+J(Z, \theta)$, the expression for $G(J(\theta, Z))$ can be approximated by $G\left(\bar{J}(Z)\right)$, so that:

$$\omega^{-1}(J, \theta, Z, 0)=G\left(\bar{J}(Z)+J(\theta)\right) \simeq G\left(\bar{J}(Z)\right)$$

Given our choice for the function G, we find:

$$\omega^{-1}(J, \theta, Z, 0) \simeq \frac{\arctan\left(\left(\frac{1}{X_{r}}-\frac{1}{X_{p}}\right)\sqrt{\bar{J}(Z)}\right)}{\sqrt{\bar{J}(Z)}}=\frac{1}{\bar{X}_{r}(Z)} \tag{46}$$

Using (46), the propagator associated to (44) $\mathcal{G}_{0}(\theta, \theta', Z, Z')$ can be directly computed. It satifies:

$$-\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta}-\omega^{-1}(J, \theta, Z, 0)\right)\mathcal{G}_{0}(\theta, \theta', Z, Z')=\delta(Z-Z')\delta(\theta-\theta')$$

and we find:

$$\mathcal{G}_{0}(\theta, \theta', Z, Z')=\delta(Z-Z')\frac{\exp(-\Lambda_{1}(Z)(\theta-\theta'))}{\Lambda(Z)}H(\theta-\theta') \tag{47}$$

where:

$$\begin{aligned}
\Lambda(Z) &= \sqrt{\frac{\pi}{2}}\sqrt{\left(\frac{1}{\sigma^{2}\bar{X}_{r}(Z)}\right)^{2}+\frac{2\alpha}{\sigma^{2}}} \\
\Lambda_{1}(Z) &= \sqrt{\left(\frac{1}{\sigma^{2}\bar{X}_{r}(Z)}\right)^{2}+\frac{2\alpha}{\sigma^{2}}}-\frac{1}{\sigma^{2}\bar{X}_{r}(Z)}
\end{aligned}$$

and H is the heaviside function:

$$\begin{aligned} H(\theta-\theta') &= 0 \text{ for } \theta-\theta'<0 \\ &= 1 \text{ for } \theta-\theta'>0 \end{aligned}$$

For the sake of simplicity, we often discard the factor $\delta(Z-Z')$ and write $\mathcal{G}_0(\theta,\theta',Z)$ for $\mathcal{G}_0(\theta,\theta',Z,Z')$. In the sequel, for the sake of simplicity, the dependency in Z of $\bar{X}_r(Z)$, $\Lambda(Z)$, $\Lambda_1(Z)$ will be implicit, so that we will write:

$$\bar{X}_r(Z) \equiv \bar{X}_r,\ \Lambda(Z) \equiv \Lambda,\ \Lambda_1(Z) \equiv \Lambda_1$$

4.2.2 Graphs expansion and two point Green functions

Having found the propagator $\mathcal{G}_0(\theta,\theta',Z)$ associated to (44), appendix 1.1.2 computes the graphs associated to the decomposition (45). The two points Green function is shown to be equal to be the inverse of the operator:

$$-\frac{1}{2}\nabla_\theta\frac{\sigma_\theta^2}{2}\nabla_\theta+\frac{1}{2}\left[\frac{\delta\left[\Psi^\dagger(\theta',Z)\nabla_\theta\left(\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\Psi(\theta,Z)\right)\right]}{\delta|\Psi|^2}\right]_{\substack{\theta'=\theta\\|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}}+\left[\frac{\delta[V(\Psi)]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}} \tag{48}$$

where:

$$\mathcal{G}_0(0,Z)=\mathcal{G}_0(\theta,\theta,Z)=\frac{\exp(-\Lambda_1(Z)(\theta-\theta'))}{\Lambda(Z)}H(\theta-\theta')$$

and $|\Psi|^2\left[\frac{\delta}{\delta|\Psi|^2}\right]$ is a shorthand for:

$$\int dZ'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\times\frac{\delta}{\delta\left(\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\right)} \tag{49}$$

4.2.3 Effective action at the lowest order

The effective order at the lowest order is derived directly from (48). Appendix 3 shows that:

$$\Gamma_0\left(\Psi^\dagger,\Psi\right)=\Psi^\dagger(\theta,Z)\left[\frac{\delta\left[S_{cl}\left(\Psi^\dagger,\Psi\right)\right]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\Psi(\theta,Z) \tag{50}$$

with:

$$\begin{aligned} S_{cl}\left(\Psi^\dagger,\Psi\right) &= -\frac{1}{2}\Psi^\dagger(\theta,Z)\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\right)\Psi(\theta,Z) \\ &+\alpha\int\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2+V(\Psi) \end{aligned} \tag{51}$$

and the brackets notation given in equation (49). Alternatively, formula (50) can also be written:

$$\begin{aligned} \Gamma_0\left(\Psi^\dagger,\Psi\right) &= -\frac{1}{2}\Psi^\dagger(\theta,Z)\left(\nabla_\theta\frac{\sigma^2}{2}\left(\nabla_\theta-\left(\omega^{-1}\left(\bar{J},Z,\mathcal{G}_0\right)+\frac{\delta\left[\omega^{-1}\left(\bar{J},Z,\mathcal{G}_0\right)\right]}{\delta\mathcal{G}_0(0,Z)}\mathcal{G}_0(\theta',\theta,Z)\right)\right)\right)_{\theta'=\theta}\Psi(\theta,Z) \\ &+\alpha\int|\Psi(\theta,Z)|^2+\Psi^\dagger(\theta,Z)\left[\frac{\delta[V(\Psi)]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\Psi(\theta,Z) \end{aligned} \tag{52}$$

where $\omega^{-1}\left(\bar{J},Z,\mathcal{G}_0\right)$ is the static inversed frequency defined as the solution of the equation:

$$\omega^{-1}\left(\bar{J}\left(Z\right),Z,\mathcal{G}_0\right)=G\left(\bar{J}\left(Z\right)+\int\frac{\kappa}{N}\frac{\omega\left(\bar{J},Z_1,\mathcal{G}_0\right)}{\omega\left(\bar{J},Z,\mathcal{G}_0\right)}\mathcal{G}_0\left(0,Z_1\right)T\left(Z,\theta,Z_1\right)dZ_1\right) \tag{53}$$

In formula (53), $\bar{J}\left(Z\right)$ is the average over time of $J\left(\theta,Z\right)$. As a consequence, $\omega^{-1}\left(\bar{J}\left(Z\right),Z,\mathcal{G}_0\right)$ solves:

$$\omega^{-1}\left(\bar{J}\left(Z\right),Z,\mathcal{G}_0\right)=G\left(\bar{J}\left(Z\right)+\int\frac{\kappa}{N}\frac{\omega^{-1}\left(\bar{J}\left(Z_1\right),Z_1,\mathcal{G}_0\right)}{\omega^{-1}\left(\bar{J}\left(Z\right),Z,\mathcal{G}_0\right)}T\left(Z,\theta,Z_1\right)\mathcal{G}_0\left(0,Z_1\right)dZ_1\right) \tag{54}$$

Once $\omega^{-1}\left(Z,\mathcal{G}_0\right)$ is known, (52) implies that the effective action at the tree-order is given by:where $V\left(\Psi\right)$ is defined in (32).

4.3 Effective action at higher orders

4.3.1 General formula

Appendix 2 shows that the effective action is a series of corrections to the classical effective action (50):

$$\Gamma\left(\Psi^{\dagger},\Psi\right)=\sum_{n=2}^{\infty}\int\left(\prod_{l=1}^{n}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)S_n\left(\left(\theta_f^{(l)},\theta_i^{(l)},Z_l\right)\right)\left(\prod_{l=1}^{n}\Psi\left(\theta_i^{(l)},Z_l\right)\right)$$

The contribution $S_n\left(\left(\theta_f^{(l)},\theta_i^{(l)},Z_l\right)\right)$ for a given n is the n points effective vertex. It is the sum of one-particle irreducible graphs (1PI) with n lines labelled by their position Z_l, $l=1,...,n$ plus their starting and ending points $\left(\theta_f^{(l)},\theta_i^{(l)}\right)$.

To build the series of graphs, we first consider the l points vertices:

$$\begin{aligned}\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)},Z_{k_i}\right)\right\}_{i=1,\dots,l}\right) &= \frac{1}{l!}\left[\frac{\delta^l\left[\int\Psi^{\dagger}\left(\theta,Z\right)\nabla_{\theta}\omega^{-1}\left(J,\theta,Z\right)\Psi\left(\theta,Z\right)dZd\theta+V\left(\Psi\right)\right]}{\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}\\ &= \frac{1}{l!}\left[\frac{\delta^l\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}\end{aligned}$$

for $l=2,...,n$ and $\left(\theta^{(k_i)},Z_{k_i}\right)\in\left\{\left(\theta_i,Z_i\right)\right\}_{i=1,\dots,n}$ with $\theta_i\in\left[\theta_i^{(i)},\theta_f^{(i)}\right]$ and $\left(\theta^{(k_i)},Z_{k_i}\right)\neq\left(\theta^{(k_j)},Z_{k_j}\right)$ for $i\neq j$.

These vertices are represented graphically by associating a point $\left(\theta,Z\right)_{\hat{V}}$ to each vertex $\hat{V}$, which differs from all the $\left(\theta^{(i)},Z_i\right)$ and $\left(\theta,Z\right)_{\hat{V}}\neq\left(\theta,Z\right)_{\hat{V}'}$ when $\hat{V}\neq\hat{V}'$. We draw l lines from $\left(\theta,Z\right)_{\hat{V}}$ ending at the points $\left(\theta^{(k_i)},Z_{k_i}\right)$.

We then consider the series of graphs $l=2,...,n$, with an arbitrary number of vertices $\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)},Z_{k_i}\right)\right\}_{i=1,\dots,l}\right)$, joining the points $\left(\theta^{(k_i)},Z_{k_i}\right)$. To each internal segment between θ and θ' at position Z, we associate a propagator $\mathcal{G}_0\left(\theta,\theta',Z\right)$ defined in (47). The effective vertex $S_n\left(\left(\theta_f^{(l)},\theta_i^{(l)},Z_l\right)\right)$ is obtained by summing the n lines-1PI graphs.

Defining:

$$\begin{aligned}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right) &\equiv S_{cl}\left(\mathcal{G}_0\left(0,Z\right)+\left|\Psi\right|^2\right)\\ &\equiv -\frac{1}{2}\left(\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta}-\omega^{-1}\left(\mathcal{G}_0\left(0,Z\right)+\left|\Psi\left(\theta,Z\right)\right|^2\right)\right)\right)\left(\mathcal{G}_0\left(\theta',\theta,Z\right)+\Psi^{\dagger}\left(\theta',Z\right)\Psi\left(\theta,Z\right)\right)\right)_{\theta'=\theta}\\ &\quad+\alpha\int\left(\mathcal{G}_0\left(0,Z\right)+\left|\Psi\left(\theta,Z\right)\right|^2\right)+V\left(\left(\mathcal{G}_0\left(0,Z\right)+\left|\Psi\left(\theta,Z\right)\right|^2\right)\right)\end{aligned} \tag{55}$$

with $V\left(\left(\mathcal{G}_0(0,Z)+|\Psi(\theta,Z)|^2\right)\right)$ given by (32):

$$V\left(\mathcal{G}_0(0,Z)+|\Psi(\theta,Z)|^2\right)=\int\left(\mathcal{G}_0(0,Z)+|\Psi(\theta,Z)|^2\right)U_0\left(\int\mathcal{G}_0(0,Z')+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\right) \tag{56}$$

the effective action writes as a series expansion (see appendix 3):

$$\begin{aligned}\Gamma\left(\Psi^{\dagger},\Psi\right) &= \hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\\ &\times\prod_{i=1}^{m}\left[\int_{\prod_l\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right]\\ &\times\frac{\prod_{l=1}^{j}\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{m!\prod_k\left(\sharp_{j,m,k}\left(\left(p_l^i\right)\right)\right)!\Lambda^{\sum_{i,l}p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)\end{aligned} \tag{57}$$

with $\left(\sharp_{j,m,k}\left(\left(p_l^i\right)\right)\right)!$ standing for the number of external lines with multiple lines of valence $k\geqslant 2$:

$$\sharp_{j,m,k}\left(\left(p_l^i\right)\right)=\sum_{l=1}^{j}\delta_{k,\sum_{i=1}^{m}p_l^i} \tag{58}$$

and where:

$$\theta_i^{(l)}<\theta^{\left(k_l^i\right)}<\theta_f^{(l)}$$

The notation $\left(p_l^i\right)$ in (58) stands for the dependency of $\sharp_{j,m,k}\left(\left(p_l^i\right)\right)$ in the whole set of indices $\left(p_l^i\right)$ with $i=1...m$ and $l=1...j$.

In the local approximation, when $\Lambda_1>1$, $\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)$ can be replaced by $\frac{1}{\Lambda_1}\delta\left(\theta_f^{(l)}-\theta_i^{(l)}\right)$ and the effective action writes:

$$\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{j\geqslant 2}\int\left[\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\sum_{\substack{m\geqslant 1,\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}\frac{\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{\sum_l p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right]}{m!\prod_k\left(\sharp_{j,m,k}\left(\left(p_l^i\right)\right)\right)!\Lambda_1^j\Lambda^{\sum_{i,l}p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta^{(l)},Z_l\right)\right)\right] \tag{59}$$

Estimation of the effective action's series expansion A series expansion for (59) can be derived. For strong fields, we have:

$$\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\simeq\frac{1}{2}\int\frac{\delta^{\sum_l p_l^i}\nabla_\theta\omega^{-1}\left(|\Psi(\theta,Z)|^2\right)}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}|\Psi(\theta,Z)|^2$$

The derivatives of $\omega^{-1}\left(J,\theta,Z,\mathcal{G}_0(0,Z)+|\Psi|^2\right)$ can be estimated as:

$$\frac{\delta^{n}\omega^{-1}\left(J,\theta,Z\right)}{\prod\limits_{i=1}^{n}\delta\left|\Psi\left(\theta-l_{i},Z_{i}\right)\right|^{2}} \simeq \frac{\exp\left(-cl_{n}-\alpha\left(\sum_{r=1}^{n-1}\left(\left(c\left(l_{r}-l_{r+1}\right)\right)^{2}-\left|Z_{r}-Z_{r+1}\right|^{2}\right)\right)\right)}{D^{n}} \times H\left(cl_{n}-\sum_{i=1}^{n-1}\left|Z_{i}-Z_{i+1}\right|\right)\prod_{i=1}^{n}\frac{\omega^{-1}\left(J,\theta-l_{i},Z_{i}\right)}{\left|\Psi\left(\theta-l_{i},Z_{l}\right)\right|^{2}} \tag{60}$$

where D is a constant (see appendix 6).

As a consequence, ordering the $\theta^{(l)}$ by $\theta^{(l)}<...<\theta^{(1)}$, we define $\theta_{i,j}=\min_{l,p_{l}^{i}\neq0}\left(\theta_{l}\right)$.

$$\begin{aligned}&\frac{\delta^{\sum_{l}p_{l}^{i}}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod\limits_{l=1}^{j}\delta^{p_{l}^{i}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}} \\ \simeq\ & -\frac{c}{2}\int\frac{\exp\left(-c\left(\theta_{i}-\theta_{i,j}\right)-\alpha\left(\sum_{l=1,p_{l}^{i}\neq0}^{j}\left(\left(c\left(\theta^{(l-1,i)}-\theta^{(l,i)}\right)\right)^{2}-\left|Z_{l-1}^{(i)}-Z_{l}^{(i)}\right|^{2}\right)\right)\right)}{D^{\sum_{l}p_{l}^{i}}} \\ &\times H\left(c\left(\theta_{i}-\theta_{i,j}\right)-\sum_{l=1,p_{l}^{i}\neq0}^{j}\left|Z_{l-1}^{(i)}-Z_{l}^{(i)}\right|\right)\prod_{l=1}^{j}\left(\frac{\omega^{-1}\left(J,\theta^{(l)},Z_{l}\right)}{\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\right)^{\sum_{i}p_{l}^{i}}\left|\Psi\left(\theta_{i},Z_{i}\right)\right|^{2}d\theta_{i}dZ_{i}\end{aligned} \tag{61}$$

We use the convention $\theta^{(l,i)}=\theta^{(l)}$, $Z_{l}^{(i)}=Z_{l}$ for $l>0$ and $\theta^{(l,0)}=\theta_{i}$, $Z_{l}^{(0)}=Z_{i}$. We can thus write (59) as:

$$\begin{aligned}&\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)-\sum_{j\geqslant1}\int\sum_{\substack{m\geqslant2,\left(p_{l}^{i}\right)_{m\times j}\\ \sum_{i}p_{l}^{i}\geqslant2}}\prod_{l=1}^{j}d\theta^{(l)}dZ_{l}\left(\frac{\omega^{-1}\left(J,\theta^{(l)},Z_{l}\right)}{\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\right)^{\sum_{i}p_{l}^{i}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2} \\ &\times\frac{\prod\limits_{i=1}^{m}\int c\exp\left(-c\left(\theta_{i}-\theta_{i,j}\right)-\alpha\left(\sum_{l=1,p_{l}^{i}\neq0}^{j}\left(\left(c\left(\theta^{(l-1,i)}-\theta^{(l,i)}\right)\right)^{2}-\left|Z_{l-1}^{(i)}-Z_{l}^{(i)}\right|^{2}\right)\right)\right)\left|\Psi\left(\theta_{i},Z_{i}\right)\right|^{2}d\theta_{i}dZ_{i}}{\left(-2\right)^{m}D^{\sum_{i,l}p_{l}^{i}}m!\prod\limits_{k}\left(\sharp_{k}\right)!\Lambda_{1}^{j}\Lambda^{\sum_{i,l}p_{l}^{i}}}\end{aligned} \tag{62}$$

where the expressions in the sum include an implicit Heaviside function:

$$H\left(-c\left(\theta_{i}-\theta_{i,j}\right)-\alpha\left(\sum_{l=1,p_{l}^{i}\neq0}^{j}\left(\left(c\left(\theta^{(l-1,i)}-\theta^{(l,i)}\right)\right)^{2}-\left|Z_{l-1}^{(i)}-Z_{l}^{(i)}\right|^{2}\right)\right)\right)$$

4.3.2 Alternative form for the effective action

An alternative form for the effective action can be derived for computational reasons (see appendix 3.3). We give its full form, and its local approximation, which is more tractable.

Full form We rewrite $\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$, up to the constant $\alpha\int\mathcal{G}_{0}\left(0,Z_{i}\right)$:

$$\begin{aligned}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right) &\simeq \int\Psi^{\dagger}\left(\theta,Z\right)\left(-\frac{1}{2}\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(\left|\Psi\left(\theta,Z\right)\right|^{2}\right)\right)\right)+\alpha+U\left(\int\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^{2}\right)\right)\Psi\left(\theta,Z\right) \\ &\equiv \int\Psi^{\dagger}\left(\theta,Z\right)L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\Psi\left(\theta,Z\right)\end{aligned}$$

In the above:

$$U\left(\int\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\right)$$

is obtained by the series expansion of:

$$V\left(\mathcal{G}_0(0,Z)+|\Psi(\theta,Z)|^2\right)$$

defined in (56), whose terms of degree 2 and higher in fields have been collected.

The effective action is given by:

$$\Gamma\left(\Psi^{\dagger},\Psi\right)=\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\Psi^{\dagger}(\theta,Z)\frac{\delta^{\sum_l p_l}\left[L\left(\Psi^{\dagger}(\theta,Z),\Psi(\theta,Z)\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}a_{j,m}(\theta,\theta_i)\Psi(\theta_i,Z)$$

$$\times\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left[\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right]$$

$$\times\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{m!\sharp_{j,m}\left(\left(p_l^i\right)\right)\Lambda^{\sum_i p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)\tag{63}$$

where the kernel $a_{j,m}(\theta,\theta_i)$ is defined as:

$$a_1(\theta,\theta_i)=\frac{\exp\left(-\Lambda_1\left(\theta-\theta_i^{(l)}\right)\right)}{\Lambda^{\sum_i p_l^i}}\prod_{i=1}^{m}\int_{\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p^i}}\frac{\delta^{p^i}}{\prod_{k_l^i=1}^{p^i}\delta\left|\Psi\left(\theta^{(k^i)},Z_m\right)\right|^2}d\theta^{\left(k^i\right)}\tag{64}$$

and

$$a_{j,m}(\theta,\theta_i)=\delta(\theta-\theta_i)+\frac{\sharp_{j+1,m+1}\left((p_l),\left(p_l^i\right)\right)}{\bar{\sharp}_{j+1,m+1}\left((p_l),\left(p_l^i\right)\right)}\frac{\exp\left(-\Lambda_1\left(\theta-\theta_i^{(l)}\right)\right)}{\Lambda^{\sum_i p_l^i}}\prod_{i=1}^{m}\int_{\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p^i}}\frac{\delta^{p^i}}{\prod_{k_l^i=1}^{p^i}\delta\left|\Psi\left(\theta^{(k^i)},Z_m\right)\right|^2}d\theta^{\left(k^i\right)}\tag{65}$$

for $j>1$. The factors $\sharp_{j,m}$ and $\frac{1}{\bar{\sharp}_{j,m}}$ are given by:

$$\sharp_{j,m}\left(\left(p_l^i\right)\right)=\prod_k\left(\bar{\sharp}_{j,m,k}\left(p_l^i\right)\right)!=\left(\sum_{l=1}^{j}\delta_{k,\sum_{i=1}^{m}p_l^i}\right)!\tag{66}$$

$$\frac{1}{\bar{\sharp}_{j,m}\left(\left(p_l^i\right)\right)}=\frac{1}{\prod_k\left(\bar{\sharp}_{j,m,k}\left(p_l^i\right)\right)!}=\sum_{p=1}^{j}\frac{1}{\prod_k\left(\sum_{l=1}^{j}\delta_{k,\sum_{i=1}^{m}p_l^i+\delta_{l,p}}\right)!}\tag{67}$$

The expression $\sharp_{j,m,k}\left(\left(p_l^i\right)\right)$ has been defined in (58). The notations $\sharp_{j+1,m+1}\left((p_l),\left(p_l^i\right)\right)$ and $\bar{\sharp}_{j+1,m+1}\left((p_l),\left(p_l^i\right)\right)$ in (65) are defined by (66) and (67) in which the multi-indices $\left(p_l^i\right)_{l=1,\ldots l}^{i=1,\ldots m}$ are replaced by the collection obtained by gathering $(p_l)_{i=1\ldots m}$ and $\left(p_l^i\right)_{l=1,\ldots l}^{i=1,\ldots m}$.

The derivatives $i=1,,,,m$ implicitly act independently on each factor:

$$\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}$$

in (63).

Local approximation In the local approximation, (63) simplifies and becomes:

$$\Gamma\left(\Psi^{\dagger},\Psi\right)=\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_l p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right) \tag{68}$$

$$\times\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta_l,Z_l\right)\right|^2}\right]\frac{\left(\prod_{l=1}^{j}\Psi\left(\theta^{(l)},Z_l\right)\right)}{m!\left(\sharp_{j,m}\left(\left(p_l^i\right)\right)\right)\Lambda^{\sum_i p_l^i}}\prod_{l=1}^{j}dZ_l d\theta_l$$

with:

$$a_{1,m}=1$$

and

$$a_{j,m}=1+\frac{\sharp_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)}{\bar{\sharp}_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)}$$

for $j>1$

4.3.3 Estimation of the series expansion for the alternative form of the effective action

The effective action can be approximated using (63). We first estimate the derivatives of $\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$. For a slowly varying potential $V\left(\Psi\right)$, we have:

$$\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{\sum_l p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\simeq\frac{\delta^{\sum_l p_l^i}\left[\int\Psi^{\dagger}\left(\theta,Z\right)\nabla_{\theta}\omega^{-1}\left(J,\theta,Z,\mathcal{G}_0\left(0,Z\right)+\left|\Psi\right|^2\right)\Psi\left(\theta,Z\right)dZd\theta\right]}{\prod_{l=1}^{j}\delta^{\sum_l p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}$$

This leads in the approximation of slowly varying background fields to the following effective action:

$$\begin{aligned}&\Gamma\left(\Psi^{\dagger},\Psi\right) \qquad (69)\\
=\;&\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left(\left(\frac{1}{2}\frac{\delta^{\sum_l p_l^i}\left(\nabla_{\theta}\omega^{-1}\left(\theta_{i+1},Z_{i+1},\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\left|\Psi\left(\theta_{i+1},Z_{i+1}\right)\right|^2\right)\right.\\
&\times\left.\left(1+\sum_{p_i}\left(\frac{\sharp_{j+1,i}\left(\left(\left(p_i\right),\left(p_l^i\right)\right)\right)}{\bar{\sharp}_{j+1,i}\left(\left(\left(p_i\right),\left(p_l^i\right)\right)\right)}\left(\frac{\delta}{\delta\left|\Psi^{\dagger}\left(\theta_i,Z_i\right)\right|^2}\right)^{p_i}\right)\right)\right)a_{j,m}\left(\int\Psi^{\dagger}\left(\theta,Z\right)\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_{\theta}\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)\right)\\
&\times\frac{\left(\prod_{l=1}^{j}\Psi\left(\theta^{(l)},Z_l\right)\right)}{(m-1)!\Lambda^{\sum_i p_l^i}}\prod_{i=1}^{m}d\theta_i dZ_i\prod_{l=1}^{j}d\theta^{(l)}dZ_l\end{aligned}$$

with the convention that $\left(\theta_{m+1},Z_{m+1}\right)=\left(\theta,Z\right)$.

Strong fields approximation For strong fields, the derivatives $\frac{\delta}{\delta|\Psi^{\dagger}(\theta_i,Z_i)|^2}$ are negligible and (69) reduces to:

$$\begin{aligned}
&\Gamma\left(\Psi^{\dagger},\Psi\right) \qquad (70)\\
=\;& \hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\int\prod_{i=1}^{m}\left(\int\Psi^{\dagger}\left(\theta_i,Z_i\right)\left\{\frac{1}{2}\frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta\omega^{-1}\left(\theta_i,Z_i,|\Psi|^2\right)\right)}{\prod_{l=1}^{j}\delta^{\sum_l p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right\}\Psi\left(\theta_i,Z_i\right)\right)\\
&\times\left(a_{j,m}\int\Psi^{\dagger}\left(\theta,Z\right)\left\{\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_\theta\omega^{-1}\left(\theta,Z,|\Psi|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right\}\Psi\left(\theta,Z\right)\right)\\
&\times\left(\prod_{l=1}^{j}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2\right)\prod_{i=1}^{m}d\theta_i dZ_i\prod_{l=1}^{j}d\theta^{(l)}dZ_l
\end{aligned}$$

The derivatives of $\omega^{-1}\left(J,\theta,Z,\mathcal{G}_0\left(0,Z\right)+|\Psi|^2\right)$ are computed in appendix 6, and given in (60). We order the $\theta^{(l)}$ by $\theta^{(l)}<...<\theta^{(1)}$, and define $\theta_i=\min_{l,p_l^i\neq 0}\left(\theta_l\right)$. We thus write (60) as:

$$\begin{aligned}
&\frac{\delta^{\sum_l p_l^i}\omega^{-1}\left(|\Psi\left(\theta,Z\right)|^2\right)}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2} \qquad (71)\\
\simeq\;& \int\frac{\exp\left(-c\left(\theta-\theta_i\right)-\alpha\left(\sum_{l=1,p_l^i\neq 0}^{j}\left(\left(c\left(\hat{\theta}^{(l-1)}-\hat{\theta}^{(l)}\right)\right)^2-\left|\hat{Z}_{l-1}-\hat{Z}_l\right|^2\right)\right)\right)}{D^{\sum_l p_l^i}}\\
&\times H\left(c\left(\theta-\theta_i\right)-\sum_{l=1,p_l^i\neq 0}^{j}\left|\hat{Z}_{l-1}-\hat{Z}_l\right|\right)\prod_{l=1}^{j}\left(\frac{\omega^{-1}\left(J,\theta^{(l)},Z_l\right)}{\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^{\sum_i p_l^i}|\Psi\left(\theta,Z\right)|^2
\end{aligned}$$

where D is a constant. We use the convention $\hat{\theta}^{(l)}=\theta^{(l)}$, $\hat{Z}_l^{(i)}=Z_l$ for $l>0$ and $\hat{\theta}^{(0)}=\theta$, $\hat{Z}_0=Z$. As a consequence, (70) writes:

$$\begin{aligned}
&\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{j\geqslant 1}\sum_{\substack{m\geqslant 2,\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}c\left[\frac{a_j}{2}\int\prod_{l=1}^{j}dl_l dZ_l\left(\frac{\omega^{-1}\left(J,\theta^{(l)},Z_l\right)}{\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^{\sum_i p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2\right. \qquad (72)\\
&\left.\times a_{j,m}\int\frac{\prod_{i=1}^{m}\left[\exp\left(-c\left(\theta-\theta_i\right)-\alpha\left(\sum_{l=1,p_l^i\neq 0}^{j}\left(\left(c\left(\hat{\theta}^{(l-1)}-\hat{\theta}^{(l)}\right)\right)^2-\left|\hat{Z}_{l-1}-\hat{Z}_l\right|^2\right)\right)\right)|\Psi\left(\theta,Z\right)|^2\right]}{\left(-2\right)^m D^{\sum_{i,l}p_l^i}m!\left(\sharp_{j,m}\left(\left(p_l^i\right)\right)\right)\Lambda_1^j\Lambda^{\sum_{i,l}p_l^i}}d\theta dZ\right]
\end{aligned}$$

Weak fields approximation For weak fields, the main contribution of the derivatives $\left(\frac{\delta}{\delta|\Psi^{\dagger}(\theta_i,Z_i)|^2}\right)^{p_i}$ is obtained for $p_i=1$ (see appendix 3.3):

$$\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{(k_l^i)},Z_l\right)\right|^2}\simeq\frac{j}{2}\frac{\delta^{\sum_l p_l^i}\left(\nabla_{\theta_1}\omega^{-1}\left(|\Psi\left(\theta_1,Z\right)|^2\right)\right)}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{(k_l^i)},Z_l\right)\right|^2}$$

and, in the local approximation, the effective action is:

$$\begin{aligned}
&\Gamma\left(\Psi^{\dagger},\Psi\right) \qquad (73)\\
=\; &\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\left(\prod_{i=1}^{m}\frac{\sharp_{j+1,m}\left(\left(p_m,\left(p_l^m\right)\right)\right)}{4\sharp_{j+1,m}\left(\left(p_i,\left(p_l^m\right)\right)\right)}\right)\frac{a_{j,m}}{2}\\
&\times\int\int\Psi^{\dagger}(\theta,Z)\frac{\delta^{\sum_{l,i}p_l^i+p_l}}{\prod_{l=1}^{j}\delta^{\sum_i p_l^i+p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\left(\frac{\left(\nabla_{\theta}\omega^{-1}\left(\theta,Z,|\Psi|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^{m+1}\Psi(\theta,Z)\prod_{l=1}^{j}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2 d\theta^{(l)}dZ_l
\end{aligned}$$

which, using the notations of the derivation of (72), leads to an expression in terms of inverse frequencies:

$$\begin{aligned}
&\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{j\geqslant 1}\sum_{\substack{m\geqslant 2,\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}} c\left[\frac{ja_j}{2}\int\prod_{l=1}^{j}dl_l dZ_l\left(\frac{\omega^{-1}\left(J,\theta^{(l)},Z_l\right)}{\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^{\sum_i p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2\right. \qquad (74)\\
&\left.\int\times\frac{\prod_{i=1}^{m}\left[\exp\left(-c\left(\theta-\theta_i\right)-\alpha\left(\sum_{l=1,p_l^i\neq 0}^{j}\left(\left(c\left(\hat{\theta}^{(l-1)}-\hat{\theta}^{(l)}\right)\right)^2-\left|\hat{Z}_{l-1}-\hat{Z}_l\right|^2\right)\right)\right)\right]}{(-2)^m D^{\sum_{i,l}p_l^i}m!\left(\sharp_{j,m}\left(\left(p_l^i\right)\right)\right)\Lambda_1^j\Lambda^{\sum_{i,l}p_l^i}}d\theta dZ\right]
\end{aligned}$$

5 Non-trivial minimum

5.1 Classical effective action

The effective action has a minimum for a wide range of parameters (see appendix 3.4). The corresponding background field decomposes into a constant part Ψ_0 and a contribution that depends on the external current. We show that for slowly varying currents $J(\theta,Z_i)$, and for $\left|\zeta^{(n)}\right|>\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0\right)$, the minimum of $\Gamma(\Psi)$ is reached for the fields $\Psi(\theta,Z)$ and $\Psi^{\dagger}(\theta,Z)$ that decompose as:

$$\Psi(\theta,Z)=\Psi_0(\theta,Z)+\delta\Psi(\theta,Z) \qquad (75)$$

and:

$$\Psi^{\dagger}(\theta,Z)=\Psi_0^{\dagger}(\theta,Z)+\delta\Psi^{\dagger}(\theta,Z) \qquad (76)$$

where $|\delta\Psi(\theta,Z)|<<|\Psi_0(\theta,Z)|$ and $\left|\delta\Psi^{\dagger}(\theta,Z)\right|<<\left|\Psi_0^{\dagger}(\theta,Z)\right|$. The fields $\Psi_0(\theta,Z)$ and $\Psi_0^{\dagger}(\theta,Z)$ minimize the potential:

$$\alpha\int\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2 dZ_i+\sum\frac{\zeta^{(n)}}{n!}\left(\mathcal{G}_0(0,Z_j)+\int\left|\Psi\left(\theta_i^{(i)}-\frac{|Z_i-Z_j|}{c},Z_j\right)\right|^2 dZ_j\right)^n$$

This minimum exists for $\alpha<<1$ and for $\left|\zeta^{(2)}\right|$ large. It is reached for a value X_0 of $\int\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2 dZ_i$, and its value is, up to an irrelevant phase:

$$\Psi_0\left(\theta^{(i)},Z_i\right)=\Psi_0^{\dagger}(\theta,Z)=\sqrt{\frac{X_0}{V}},$$

where V is the volume of the thread.

The expression for $\delta\Psi(\theta,Z)$ and $\delta\Psi^{\dagger}(\theta,Z)$ arising in equations (75) and (76) are found in appendix 3.4, which shows that the second-order expansion of the effective action is:

$$\begin{aligned}\Gamma(\Psi) &= -\frac{1}{2}\delta\Psi^{\dagger}(\theta,Z)\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+X_{0}+\sqrt{X_{0}}\left(\delta\Psi^{\dagger}+\delta\Psi\right)\right)\right)\right)\Psi_{0}(\theta,Z) \qquad (77)\\ &-\frac{1}{2}\delta\Psi^{\dagger}(\theta,Z)\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)\right)\right)\delta\Psi(\theta,Z)\\ &+\frac{1}{2}\delta\Psi^{\dagger}(\theta,Z)U''(X_{0})\delta\Psi(\theta,Z)\end{aligned}$$

and that the first-order equations for $\delta\Psi^{\dagger}(\theta,Z)$ and $\delta\Psi(\theta,Z)$ are:

$$\begin{aligned}\delta\Psi^{\dagger} &= 0\\ \delta\Psi(\theta,Z) &= \left(\frac{\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)\right)\right)}{U''(X_{0})-\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)\right)\right)}\right)\Psi_{0}(\theta,Z)\end{aligned} \qquad (78)$$

Setting $V=1$ yields, in first approximation:

$$\delta\Psi(\theta,Z)\simeq-\frac{\nabla_{\theta}\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)}{U''(X_{0})}X_{0} \qquad (79)$$

This relation is sufficient to derive the next section's frequencies equations, but can however be used to find $\delta\Psi(\theta,Z)$, at our order of approximation (see appendix 3.4). In the local approximation and for slowly varying currents, we show that the minimization of action (55) yields:

$$\begin{aligned}\delta\Psi(\theta,Z) &= \left(G^{-1}\left(-\frac{U''(X_{0})}{X_{0}}\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_{0}(Z_{1})+\sqrt{X_{0}}\right)}+d\right)\right)\right)-J(\theta,Z)\right) \qquad (80)\\ &\times\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_{0}(Z_{1})+\sqrt{X_{0}}\right)}+d\right)\right)\end{aligned}$$

with:

$$H(Y)=\int\frac{dY}{G^{-1}\left(-\frac{U''(X_{0})}{X_{0}}\exp Y\right)-J(\theta,Z)}$$

and:

$$\Gamma=\int\frac{\kappa}{N}\frac{|Z-Z_{1}|}{c}T\left(Z,\theta,Z_{1},\theta-\frac{|Z-Z_{1}|}{c}\right)dZ_{1}$$

The constant d is chosen so that $\lim_{\theta\to\infty}\delta\Psi(\theta,Z)=0$.

The field $\Psi\left(\theta^{(j)},Z_{j}\right)$ is the - phase-dependent - background field. It is null in the trivial phase, so that the effective action is the "classical" one. In a non-trivial phase, $\Psi\left(\theta^{(j)},Z_{j}\right)$ is not null and may be time-dependent. It describes the accumulation of currents or signals that shapes the long-term dynamics of frequencies. Incidentally, we note that a non-trivial minimum that depends on the system parameters should allow for phase transition in the system of frequencies. This question is left for further work.

5.2 Including higher order corrections

Equation (72) yields the perturbative corrections that modify the classical effective action and its minimum. These corrections modify equations (78) and (79) by shifting $U''(X_{0})\to U''(X_{0})-C(X_{0})$ (see appendix 3.4 for the expression of $C(X_{0})$), which in turn modifies the solution (80):

$$\begin{aligned}\delta\Psi(\theta,Z) &= \left(G^{-1}\left(-\frac{U''(X_{0})-C(X_{0})}{X_{0}}\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_{0}(Z_{1})+\sqrt{X_{0}}\right)}+d\right)\right)\right)-J(\theta,Z)\right) \qquad (81)\\ &\times\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_{0}(Z_{1})+\sqrt{X_{0}}\right)}+d\right)\right)\end{aligned}$$

where the constant d is set to ensure $\lim_{\theta\rightarrow\infty}\delta\Psi(\theta,Z)=0$.

6 Equation for frequencies: General form

6.1 Principle

To find the frequencies, we use the form (63) for the effective action. Writing $\Gamma\left(\Psi^{\dagger},\Psi\right)$ in the local approximation as:

$$\Gamma\left(\Psi^{\dagger},\Psi\right)\simeq\int\Psi^{\dagger}(\theta,Z)\left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)\right)\delta\left(\theta_{f}-\theta_{i}\right)+\Omega(\theta,Z)\right)\Psi(\theta,Z) \tag{82}$$

with:

$$\begin{aligned}\Omega(\theta,Z) &= \int\sum_{\substack{j\geqslant1\\m\geqslant1}}\sum_{\substack{\left(p_{l}^{i}\right)_{m\times j}\\p_{l}+\sum_{i}p_{l}^{i}\geqslant2}}\frac{\delta^{\sum_{l}p_{l}}\left[L\left(\Psi^{\dagger}(\theta,Z),\Psi(\theta,Z)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}}\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\Psi(\theta,Z) \\ &\times\frac{a_{j}}{j!}\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_{f}^{(l)},Z_{l}\right)\right)\prod_{i=1}^{m}\left[\frac{\delta^{\sum_{l}p_{l}^{i}}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{i}}\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\right]\left(\prod_{l=1}^{j}\Psi\left(\theta_{i}^{(l)},Z_{l}\right)\right)\end{aligned} \tag{83}$$

The effective frequency can be identified as:

$$\nabla_{\theta}\omega_{e}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)=\nabla_{\theta}\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)+\Omega(\theta,Z)$$

that is:

$$\omega_{e}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)=\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)+\int^{\theta}\Omega(\theta,Z) \tag{84}$$

where $\omega\left(J(\theta),\theta,Z,\bar{\mathcal{G}}_{0}+|\Psi|^{2}\right)$ is the solution of:

$$\begin{aligned}\omega^{-1}(\theta,Z) &= G\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_{1})\frac{\omega\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)}{\omega(\theta,Z)}\right. \\ &\left.\times W\left(\frac{\omega(\theta,Z)}{\omega\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)}\right)\left(\bar{\mathcal{G}}_{0}(0,Z_{1})+\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}\right)dZ_{1}\right)\end{aligned} \tag{85}$$

Using the form of $\Psi\left(\theta^{(j)},Z_{j}\right)=\Psi_{0}\left(\theta^{(j)},Z_{j}\right)+\delta\Psi(\theta,Z)$ derived in section 5, we find an expression for $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)$.

The second term $\int^{\theta}\Omega(\theta,Z)$ in (84) represents corrections due to the interactions. Using (83), we can find its expression as a series expansion in terms of frequencies and field. In the next two paragraph, we limit ourselves to the cases of strong and weak field approximation, respectively.

6.2 Strong field approximation

Using (70), the strong field approximation is:

$$\omega_e^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)=\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right) \tag{86}$$
$$+\int^{\theta}\sum_{\substack{j\geqslant1\\m\geqslant1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\p_l+\sum_i p_l^i\geqslant2}}\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_\theta\omega^{-1}\left(|\Psi(\theta,Z)|^2\right)\right)}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}$$
$$\times\int\frac{a_j}{j!}\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\omega^{-1}\left(|\Psi(\theta,Z)|^2\right)}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}|\Psi(\theta,Z)|^2\right]\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)$$

We will compute in section 8.4 the lowest order terms of the correction terms in (86) and inspect their impact on the frequencies dynamics.

6.3 Weak field approximation

Using (73), the weak field approximation is:

$$\omega_e^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)=\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right) \tag{87}$$
$$+\int^{\theta}\sum_{\substack{j\geqslant1\\m\geqslant1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\p_l+\sum_i p_l^i\geqslant2}}\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_\theta\omega^{-1}\left(|\Psi(\theta,Z)|^2\right)\right)}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}$$
$$\times\int\frac{a_j}{j!}\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left[\frac{j}{2}\frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta\omega^{-1}\left(|\Psi(\theta,Z)|^2\right)\right)}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\right]\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)$$

7 Static equilibrium for frequencies

Discarding the corrections terms $\int^{\theta}\Omega(\theta,Z)$, a static solution of (84) can be found for a constant background $\Psi\left(\theta^{(1)},Z_1\right)\simeq\Psi_0(Z_1)$ and constant current, i.e. $J=\bar{J}$, $\omega(\theta,Z)=\omega(Z)$. For a static solution, it implies $\left(\Psi^{\dagger}\mathcal{G}_0^{-1}\Psi\right)=0$, or equivalently: $\delta\Psi(\theta,Z)=\nabla_\theta\omega^{-1}(J(\theta),\theta,Z,\mathcal{G}_0+X_0)=0$, and:

$$\omega_e^{-1}(J(\theta),\theta,Z)=\omega^{-1}\left(\bar{J},Z,\mathcal{G}_0(0,Z)+X_0\right)$$

where $\omega\left(\bar{J}(Z),\mathcal{G}_0(0,Z_i)+X_0\right)$ is solution of:

$$\omega(Z)=F\left(\bar{J}+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega(Z_1)}{\omega(Z)}W\left(\frac{\omega(Z)}{\omega(Z_1)}\right)\bar{\mathcal{G}}_0(0,Z_i)\,dZ_1\right) \tag{88}$$

and:

$$\bar{\mathcal{G}}_0(0,Z_i)\simeq\mathcal{G}_0(0,Z_i)+X_0 \tag{89}$$

Moreover, in the absence of external sources, i.e. for $J(\theta,Z)=0$, the solution of (88) can be written $\omega_0(Z)$, which satisfies:

$$\omega_0(Z)=F\left(\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega(Z_1)}{\omega(Z)}W\left(\frac{\omega(Z)}{\omega(Z_1)}\right)\bar{\mathcal{G}}_0(0,Z_i)\,dZ_1\right) \tag{90}$$

where $\frac{1}{N}\int dZ_1$ is normalized to 1.

8 Differential equation for frequencies in the local approximation

A local approximation of (84) under some position-independent static equilibrium can be derived. It will be later generalized to a position-dependent equilibrium.

8.1 Assumptions

First, assume that the background field:

$$\Psi\left(\theta^{(j)}, Z_j\right) = \Psi_0\left(Z_j\right) + \delta\Psi\left(\theta^{(j)}, Z_j\right)$$

satisfies:

$$\left|\delta\Psi\left(\theta^{(j)}, Z_j\right)\right| << \left|\Psi_0\left(Z_j\right)\right|$$

Then, considering a translation-independent transfer function, i.e. $T\left(Z, Z_1\right) = T\left(Z - Z_1\right)$ with $d >> 1$, and neglecting border effects, equation (90) simplifies and yields a constant solution:

$$\omega_0 = F\left(\frac{TW\left(1\right)}{\bar{\Lambda}}\right) \tag{91}$$

where:

$$\begin{aligned} T &= \frac{\kappa}{N}\int T\left(Z, Z_1\right) dZ_1 \\ \frac{T}{\bar{\Lambda}} &= \frac{\kappa}{N}\int T\left(Z, Z_1\right)\mathcal{G}_0\left(0, Z_1\right) dZ_1 \end{aligned}$$

Ultimately, we assume that the transfer functions are symmetric, that is:

$$T\left(Z, Z_1\right) = T\left(Z_1, Z\right) \tag{92}$$

8.2 Local equation for frequencies

Note that, given (30) and (92), we have:

$$\begin{aligned} &\frac{\partial}{\partial\omega\left(\theta, Z\right)}\left(\frac{\kappa}{N}\int W\left(\frac{\omega}{\omega_1}\right) dZ_1\right)_{\omega_1\left(\theta_1, Z_1\right) = \omega\left(\theta, Z\right)} \\ &+\frac{\partial}{\partial\omega_1\left(\theta, Z\right)}\left(\frac{\kappa}{N}\int W\left(\frac{\omega}{\omega_1}\right) dZ_1\right)_{\omega_1\left(\theta_1, Z_1\right) = \omega\left(\theta, Z\right)} = \frac{\kappa}{N}\int\frac{\left(W'\left(1\right) - W'\left(1\right)\right)}{\omega\left(\theta, Z\right)} dZ_1 = 0 \end{aligned}$$

We can find a local approximation of (84) if we expand $\omega\left(J\left(\theta\right), \theta, Z, \mathcal{G}_0 + \left|\Psi\right|^2\right)$ to the second-order in $Z - Z_1$, and consider the other terms in the right-hand side of (84) as corrections. The equation for $\omega\left(J\left(\theta\right), \theta, Z, \mathcal{G}_0 + \left|\Psi\right|^2\right)$ is:

$$\begin{aligned} &F^{-1}\left(\omega\left(J\left(\theta\right), \theta\right)\right) \\ = &\left(J\left(\theta\right) + \int\frac{\kappa T\left(Z, Z_1\right)}{N}\frac{\omega\left(\theta - \frac{\left|Z - Z_1\right|}{c}, Z_1\right)}{\omega\left(\theta, Z\right)} W\left(\frac{\omega\left(\theta, Z\right)}{\omega\left(\theta - \frac{\left|Z - Z_1\right|}{c}, Z_1\right)}\right)\right. \\ &\left.\times\left(\mathcal{G}_0\left(0, Z_1\right) + \left|\Psi_0 + \delta\Psi\left(\theta - \frac{\left|Z - Z_1\right|}{c}, Z_1\right)\right|^2\right) dZ_1\right) \end{aligned} \tag{93}$$

We then expand $\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)$ around $\omega(\theta,Z)$ to the second-order in $Z-Z_1$ and compute the integrals, which yields for the right-hand side of (93):

$$\begin{aligned}&J(\theta)+\int\frac{\kappa T(Z,Z_1)}{N}\frac{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(\theta,Z)}W\left(\frac{\omega(\theta,Z)}{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\\&\times\left(\mathcal{G}_0(0,Z_1)+\left|\Psi_0(Z_1)+\delta\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\\\simeq\;&J(\theta)+\frac{TW(1)}{\bar{\Lambda}}+\frac{\hat{f}_1\nabla_\theta\omega(\theta,Z)}{\omega(\theta,Z)}+\frac{\hat{f}_3\nabla_\theta^2\omega(\theta,Z)}{\omega(\theta,Z)}+c^2\frac{\hat{f}_3\nabla_Z^2\omega(\theta,Z)}{\omega(\theta,Z)}+T\Psi_0\delta\Psi(\theta,Z)\end{aligned}$$

where we defined:

$$\begin{aligned}\hat{f}_1&=\frac{W'(1)-W(1)}{c}\Gamma_1,\;\hat{f}_3=\frac{(W(1)-W'(1))\Gamma_2}{c^2}\\\Gamma_1&=\frac{\kappa}{NX_r}\int|Z-Z_1|T(Z,Z_1)\bar{\mathcal{G}}_0(0,Z_1)dZ_1\\\Gamma_2&=\frac{\kappa}{2NX_r}\int(Z-Z_1)^2T(Z,Z_1)\bar{\mathcal{G}}_0(0,Z_1)dZ_1\end{aligned}\tag{94}$$

and:

$$T\Psi_0\delta\Psi(\theta,Z)=\int\frac{\kappa T(Z,Z_1)}{N}\Psi_0(Z_1)\delta\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)dZ_1$$

Using (91), equation (93) then becomes:

$$F^{-1}(\omega(J(\theta),\theta))-F^{-1}(\omega_0)=J(\theta)+\frac{\hat{f}_1\nabla_\theta\omega(\theta,Z)}{\omega(\theta,Z)}+\frac{\hat{f}_3\nabla_\theta^2\omega(\theta,Z)}{\omega(\theta,Z)}+c^2\hat{f}_3\frac{\nabla_Z^2\omega(\theta,Z)}{\omega(\theta,Z)}+T\Psi_0\delta\Psi(\theta,Z)\tag{95}$$

Using also the local linear approximation for $\delta\Psi(\theta,Z)$ derived in appendix 4.4.2:

$$\begin{aligned}\delta\Psi(\theta,Z)&\simeq\frac{\nabla_\theta\omega\left(\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}{U''(X_0)\,\omega^2\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}\Psi_0\\&\simeq\frac{\nabla_\theta\omega(\theta,Z,\mathcal{G}_0)}{U''(X_0)\,\omega^2(J(\theta),\theta,Z,\mathcal{G}_0)}\Psi_0\end{aligned}\tag{96}$$

leads to:

$$\begin{aligned}T\delta\Psi(\theta,Z)&\simeq\delta\Psi(\theta,Z)-\Gamma_1\nabla_\theta\delta\Psi(\theta,Z)\\&\simeq N_1\nabla_\theta\omega(\theta,Z,\mathcal{G}_0)-N_2\nabla_\theta\omega(\theta,Z,\mathcal{G}_0)\end{aligned}$$

with:

$$\begin{aligned}N_1&=\frac{\Psi_0(Z)}{U''(X_0)\,\omega^2(J(\theta),\theta,Z,\mathcal{G}_0)}\\N_2&=\frac{\Gamma_1\Psi_0(Z)}{U''(X_0)\,\omega^2(J(\theta),\theta,Z,\mathcal{G}_0)}\end{aligned}$$

We assume that F^{-1} is slowly varying, so that:

$$F^{-1}(\omega(J(\theta),\theta))-F^{-1}(\omega_0)\simeq\Gamma_0(\omega(J(\theta),\theta)-\omega_0)$$

with[1]:

$$f=\left(F^{-1}\right)'\left(\frac{\kappa}{N}\int T(Z,Z_1)W(1)\,dZ_1\bar{\mathcal{G}}_0(0,Z_1)\right)$$

[1]Given our assumption that F is an increasing function, $f>0$.

and define:

$$\Omega(\theta, Z) = \omega(\theta, Z) - \omega_0$$

As a result, the expansion of (95) for a non-static current is then:

$$f\Omega(\theta, Z) = J(\theta) + \left(\frac{\hat{f}_1}{\omega(\theta, Z)} + N_1\right)\nabla_\theta \Omega(\theta, Z) + \left(\frac{\hat{f}_3}{\omega(\theta, Z)} - N_2\right)\nabla_\theta^2 \Omega(\theta, Z) + \frac{c^2 \hat{f}_3}{\omega(\theta, Z)}\nabla_Z^2 \Omega(\theta, Z) \quad (97)$$

8.3 Stable traveling waves solutions of (97)

When $J(\theta)$ is set to 0, equation (97) has some stable non linear oscillatory solutions, for a certain range of the parameters. As a consequence, equation (97) behaves locally as a wave equation, provided that $\hat{f}_3 > 0$, $\frac{\hat{f}_3}{\omega(\theta,Z)} - N_2 < 0$, and $\omega(\theta, Z)$ varies slowly. An approximative solution of the type:

$$\omega(\theta, Z) = \exp(ik_0\theta - ikZ)$$

can be found by writing:

$$k_0^2 + i\frac{\frac{\hat{f}_1}{\bar{\omega}(\theta,Z)} + N_1}{N_2 - \frac{\hat{f}_3}{\bar{\omega}(\theta,Z)}}k_0 - \frac{\frac{c^2\hat{f}_3}{\bar{\omega}(\theta,Z)}k^2 + f}{N_2 - \frac{\hat{f}_3}{\bar{\omega}(\theta,Z)}} = 0 \quad (98)$$

and:

$$k_0 = -\frac{i}{2}\left(\frac{\hat{f}_1}{\bar{\omega}(\theta, Z)} + N_1\right) \pm \sqrt{\frac{\frac{c^2\hat{f}_3}{\bar{\omega}(\theta,Z)}k^2 + f}{N_2 - \frac{\hat{f}_3}{\bar{\omega}(\theta,Z)}} - \frac{1}{4}\left(\frac{\frac{\hat{f}_1}{\bar{\omega}(\theta,Z)} + N_1}{N_2 - \frac{\hat{f}_3}{\bar{\omega}(\theta,Z)}}\right)^2}$$

where $\bar{\omega}(\theta, Z)$ is the average of $\omega(\theta, Z)$ in some range of time.

The approximative solution is oscillatory and explosive when the discriminant of (98) is positive and in a range for $\omega(\theta, Z)$, such that $\frac{\hat{f}_1}{\omega(\theta,Z)} + N_1 > 0$. The solution is oscillatory and dampening for $\frac{\hat{f}_1}{\omega(\theta,Z)} + N_1 < 0$.

Let define ω_1 the value of $\omega(\theta, Z)$ such that $\frac{\hat{f}_1}{\omega_1} + N_1 = 0$. For $\omega(\theta, Z) > \omega_1$ the solution of (97) presents an increasing amplitude, and for $\omega(\theta, Z) > \omega_1$ the solution of (97) is decreasing in amplitude.

If $\omega_1 > \omega_0$, then ω_0 is a stable point since it belongs to the domain in which $\frac{\hat{f}_1}{\omega(\theta,Z)} + N_1 < 0$. Oscillatory patterns will dampen towards ω_0.

If $\omega_1 < \omega_0$, the frequency ω_0 is an unstable point. However, the change in sign of $\frac{\hat{f}_1}{\omega(\theta,Z)} + N_1$ for some ranges in the parameters, induces an oscillatory pattern around ω_0. Actually, when the oscillation of A is of quite constant amplitute, the time $T_{<\omega_1}$ spent by the system below ω_1 is proportional to $\arccos\left(\frac{\omega_0-\omega_1}{A}\right)$, and the time $T_{>\omega_1}$ spent by the system above ω_1 is proportional to $1 - \arccos\left(\frac{\omega_0-\omega_1}{A}\right)$. For ω_1 large enough and during the time $T_{<\omega_1}$ the system amplitude is multiplied by a term of order $\exp(-\omega_1 T_{<\omega_1})$, whereas during the time $T_{>\omega_1}$ the system amplitude is multiplied by a term of order $\exp\left(\frac{1}{N_1}T_{>\omega_1}\right)$. Since $\omega_0 - \omega_1 > 0$, the relation $T_{<\omega_1} < T_{>\omega_1}$ is always true. However, the overall factor $\exp\left(-\omega_1 T_{<\omega_1} + \frac{1}{N_1}T_{>\omega_1}\right)$ may, depending on the system's parameters and on $T_{<\omega_1}$, be lower or grater than 1. Moreover its magnitude depends on the dynamics, since the amplitude A that determines $T_{<\omega_1}$ is itself time-dependent.

When $\omega(0, Z) > \omega_1$, A increases, and the time $T_{<\omega_1}$ increases with A. For some values of the parameters, the dampening factor $\exp(-\omega_1 T_{<\omega_1})$ cumulated during time $T_{<\omega_1}$ becomes dominant with respect to the increasing factor $\exp\left(\frac{1}{N_1}T_{>\omega_1}\right)$. As a consequence, $\exp\left(-\omega_1 T_{<\omega_1} + \frac{1}{N_1}T_{>\omega_1}\right) > 1$. The dynamic pattern turns from explosive to dampened. Thus, the average amplitude A decreases, and $T_{<\omega_1}$ diminishes. At some point, $\exp\left(-\omega_1 T_{<\omega_1} + \frac{1}{N_1}T_{>\omega_1}\right)$ becomes greater than 1 and the amplitude increases again. The resulting dynamics thus presents stable oscillations that are irregular in amplitude. The system does not converge toward ω_0 which remains unstable, but presents the characteristic of some non linear travelling wave.

Note that this result is more general than the one obtained in the linear approximation. In our context, the possibility of travelling stable oscillation is obtained for a whole range of parameters whereas the linear approximation implies more restrictive condition. Actually, in the linear approximation, equation (97) reduces to:

$$f\Omega(\theta,Z)=J(\theta)+\left(\frac{\hat{f}_1}{\omega_0}+N_1\right)\nabla_\theta\Omega(\theta,Z)+\left(\frac{\hat{f}_3}{\omega_0}-N_2\right)\nabla_\theta^2\Omega(\theta,Z)+\frac{c^2\hat{f}_3}{\omega_0}\nabla_Z^2\Omega(\theta,Z) \tag{99}$$

and the condition for some stable linear travelling wave is obtained only for one value:

$$\frac{\hat{f}_1}{\omega_0}+N_1=0 \tag{100}$$

However, once the possibility of travelling wave is understood, one can replace (97) by its linear version (99) where (100) is assumed to be satisfied.

8.4 Interaction corrections to the wave equation (97)

Equation (84) yields the corrective terms to $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$. We focus on the weak field approximation (87) to ensure corrections of small magnitude. Since $p_l+\sum_i p_l^i\geqslant 2$, the lowest order correction is for $m=1$ and $p_l+\sum_i p_l^i\geqslant 2=2$, and appendix 4.3 shows that:

$$\omega_e^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)=\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)+Z \tag{101}$$

where:

$$\begin{aligned}Z &= \int^\theta d\theta\sum_{\substack{j\geqslant1\\m\geqslant1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\p_l+\sum_i p_l^i\geqslant2}}\left(\prod_{i=1}^m\frac{\sharp_{j+1,m}\left(\left(p_m,\left(p_l^m\right)\right)\right)}{4\sharp_{j+1,m}\left(\left(p_i,\left(p_l^m\right)\right)\right)}\right)\frac{a_{j,m}}{2} \\ &\times\int\prod_{i=1}^m\left\{\frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta\omega^{-1}\left(\theta,Z,|\Psi|^2\right)\right)}{\prod_{l=1}^j\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right\}\frac{\delta^{\sum_l p_l}\left(\nabla_\theta\omega^{-1}\left(\theta,Z,|\Psi|^2\right)\right)}{\prod_{l=1}^j\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\prod_{l=1}^j\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2d\theta^{(l)}dZ_l\end{aligned} \tag{102}$$

This series take into account the interaction between the frequencies and the background field. To find detailed results, we limit ourselves to the lowest order corrections. We show in appendix 3, that these corrections have the form:

$$\begin{aligned}\omega_e^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right) &= \omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right) \\ &-\frac{f}{4\left(\frac{\hat{f}_3}{\omega(\theta,Z)}-N_2\right)}\int\int^\theta\left(\frac{\delta\left(\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^2 \\ &+\int\frac{1}{4}\nabla_\theta\left(\frac{\delta\left(\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^2\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2 \\ &+\frac{\frac{\hat{f}_1}{\omega(\theta,Z)}+N_1}{4\left(\frac{\hat{f}_3}{\omega(\theta,Z)}-N_2\right)}\int\left(\frac{\delta\left(\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^2\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2\end{aligned} \tag{103}$$

In (103), the second and the third contributions:

$$-\frac{f}{4\left(\frac{\hat{f}_3}{\omega(\theta,Z)}-N_2\right)}\int\int^{\theta}\left(\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2}\right)^2\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2 \tag{104}$$
$$+\int\frac{1}{4}\nabla_{\theta}\left(\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2}\right)^2\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2$$

describe the influence of the collective state defined by $\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2$ on the frequency at position Z and time θ. Under our previous assumption that $\frac{\hat{f}_3}{\omega(\theta,Z)}-N_2<0$, and given that $f>0$, the first term in (104) is positive, and thus, this term reduces $\omega\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)$. The higher the sensitivity $\frac{\delta\left(\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2}$ of the frequency to the collective state, the more the frequency of the wave is reduced. The effect of this smooting is cumulative in time, as shown by the integral over time arising in this term. The second contribution in (104) amplifies this smoothing. Actually, this term is positive when $\frac{\delta\left(\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2}$, i.e. the sensitivity of frequency to the background field, increases in absolute value. As a consequence, it reduces the oscillations of $\omega\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)$ when the frequency's dependency in the background field increases at position Z and time θ.

The fourth term in (103):

$$\frac{\frac{\hat{f}_1}{\omega(\theta,Z)}+N_1}{4\left(\frac{\hat{f}_3}{\omega(\theta,Z)}-N_2\right)}\int\left(\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2}\right)^2\left|\Psi\left(\theta^{(l)},Z_{_l}\right)\right|^2$$

reinforces the mechanism of oscillation stabilization described in section 8.3. It has the sign of $-\left(\frac{\hat{f}_1}{\omega(\theta,Z)}+N_1\right)$, given our assumption $\frac{\hat{f}_3}{\omega(\theta,Z)}-N_2<0$ ensuring oscillatory behavior of $\omega\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)$. Thus, the correction to $\omega\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)$ induced by this term has the sign of $\left(\frac{\hat{f}_1}{\omega(\theta,Z)}+N_1\right)$: for $\frac{\hat{f}_1}{\omega(\theta,Z)}+N_1>0$, the approximative solution is oscillatory and explosive. Thus, the correction amplifies the oscillations of $\omega\left(\theta,Z\right)$ and the stabilization mechanism applies. For $\omega\left(\theta,Z\right)$ such that $\frac{\hat{f}_1}{\omega(\theta,Z)}+N_1<0$, the correction term turns negative and further decreases $\omega\left(\theta,Z\right)$.

The series of higher corrections is computed in appendix 4.3. It shows that, in the local approximation, the frequencies can be described by a wave equation whose form depends on the stabilization potential and the evolution of the background itself.

8.5 Some extensions

8.5.1 Multiple components field

A multiple-components field which describes excitatory vs inhibitory currents leads to frequencies equations that are similar to (43) when interaction corrections are neglected :

$$\begin{aligned}\omega_i\left(\theta,Z\right) &= F_i\left(J\left(\theta\right)+\frac{\kappa}{N}\int T\left(Z,Z_1\right)\frac{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega_i\left(\theta,Z\right)}G^{ij}\right. \\ &\left.\times W\left(\frac{\omega_i\left(\theta,Z\right)}{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\bar{\mathcal{G}}_{0j}\left(0,Z_1\right)+\left|\Psi_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned} \tag{105}$$

Similar computations to those leading to (97) yield:

$$f\Omega(\theta,Z)=J_i(\theta)+\left(\frac{\hat{f}_{1ij}}{\omega_i(\theta,Z)}+N_{1i}\delta_{ij}\right)\nabla_\theta\Omega_j(\theta,Z)+\left(\frac{\hat{f}_{3ij}}{\omega_i(\theta,Z)}-N_{2i}\right)\nabla_\theta^2\Omega_j(\theta,Z)+\frac{c^2\hat{f}_{3ij}}{\omega(\theta,Z)}\nabla_Z^2\Omega_j(\theta,Z) \tag{106}$$

where the sum over j is implicit and with:

$$\begin{aligned} N_{1i} &= \frac{\Psi_{0i}(Z)}{U''(X_0)\,\omega_i^2(J(\theta),\theta,Z,\mathcal{G}_0)} \\ N_{2i} &= \frac{\Gamma_{1i}\Psi_{0i}(Z)}{U''(X_0)\,\omega_i^2(J(\theta),\theta,Z,\mathcal{G}_0)} \end{aligned} \tag{107}$$

$$\begin{aligned} \hat{f}_{1ij} &= \frac{W'\left(\frac{\omega_i(\theta,Z)}{\omega_j(\theta,Z)}\right)-W\left(\frac{\omega_i(\theta,Z)}{\omega_j(\theta,Z)}\right)}{c}\Gamma_{1ij},\ \hat{f}_{3ij}=\frac{\left(W\left(\frac{\omega_i(\theta,Z)}{\omega_j(\theta,Z)}\right)-W'\left(\frac{\omega_i(\theta,Z)}{\omega_j(\theta,Z)}\right)\right)\Gamma_{2ij}}{c^2} \\ \Gamma_{1ij} &= \frac{\kappa}{NX_r}G^{ij}\int|Z-Z_1|\,T(Z,Z_1)\,\bar{\mathcal{G}}_{0j}(0,Z_1)\,dZ_1 \\ \Gamma_{2ij} &= \frac{\kappa}{2NX_r}G^{ij}\int(Z-Z_1)^2\,T(Z,Z_1)\,\bar{\mathcal{G}}_{0j}(0,Z_1)\,dZ_1 \end{aligned} \tag{108}$$

Equation (106) describes the interactions of several non linear traveling waves.

8.5.2 Non constant background frequency

In the above, we considered translation-invariant transfer functions. Although correct in first approximation, this hypothesis does not hold in general. For instance, it may be invalidated by finite volume of the system or border conditions. Moreover, since the whole system depends on the collective state, one may expect that endogeneizing the transfer functions induce the emergence of states with position-dependent transfer functions. A mechanism for this emergence is described in section 8.6.

We will thus consider transfer functions of the form $T(Z,Z_1)$. To make things simpler, we dismiss the corrections to the frequencies due to the potential and the background field and focus on the linear approximation (99) of (97).

The derivation of the linearized expansion of (84) around $\omega_0(Z)$ is similar to that of (97), but now yields a wave equation in an inhomogeneous medium:

$$\sigma_\theta^2\nabla_\theta^2\hat{\Omega}(\theta,Z)=g_0(Z)\,\hat{\Omega}(\theta,Z)-g_1(Z)\,\nabla_\theta\hat{\Omega}(\theta,Z)+g_2(Z)\,\nabla_Z^2\hat{\Omega}(\theta,Z) \tag{109}$$

where we defined:

$$
\begin{aligned}
\hat{\Omega}(\theta, Z) &= \frac{\Omega(\theta, Z)}{\omega_0(Z)} \\
g_1(Z) &= \frac{\Gamma_1'(Z) - \Gamma_1(Z)}{c\sigma_\theta^2 + \frac{\Gamma_2'(Z)-\Gamma_2(Z)}{c}} \\
g_2(Z) &= \frac{\Gamma_2'(Z) - \Gamma_2(Z)}{\sigma_\theta^2 + \frac{\Gamma_2'(Z)-\Gamma_2(Z)}{c^2}} \\
\Gamma_1(Z) &= \frac{\kappa}{NX_r} \int \frac{|Z - Z_1| T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z)\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0(Z) \\
\Gamma_1' &= \frac{\kappa}{NX_r} \int \frac{|Z - Z_1| T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W'\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z)\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0 \\
\Gamma_2 &= \frac{\kappa}{2NX_r} \frac{\int (Z - Z_1)^2 T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z)\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0 \\
\Gamma_2'(Z) &= \frac{\kappa}{2NX_r} \frac{\int (Z - Z_1)^2 T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W'\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\omega_0(Z)\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0(Z) \\
\Gamma_0(Z) &= G'\left(\frac{\kappa}{N} \int \frac{T(Z, Z_1) W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}}\right)
\end{aligned}
$$

8.5.3 Arbitrary transfer functions

We can derive a straightforward generalization of (97) by considering anisotropic transfer functions. So far, we have assumed that:

$$
\int (Z - Z_1)_i (Z - Z_1)_j T(Z, Z_1) \frac{\omega_0(Z_1)}{\omega_0(Z)} W\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) dZ_1 = \delta_{i,j}
$$

where $\delta_{i,j}$ is the Kronecker symbol. Relaxing this condition, we can replace $f_2(Z) \to f_2^{ij}(Z)$, $g_2(Z) \to g_2^{ij}(Z) = \frac{f_2^{ij}(Z)}{1+f_3(Z)}$. Equation (97) becomes:

$$
\nabla_\theta^2 \Omega(\theta, Z) = g_0(Z)\Omega(\theta, Z) + g_1(Z)\nabla_\theta \Omega(\theta, Z) + g_2^{ij}(Z)\nabla_{Z_i}\nabla_{Z_j}\Omega(\theta, Z) \tag{110}
$$

for distributions:

$$
\begin{aligned}
f_2(Z) &= (\omega_0 W'(1) - W(1)) \Gamma_2^{ij} \\
\Gamma_2^{ij} &= \frac{\kappa}{2NX_r} \frac{\int (Z - Z_1)_i (Z - Z_1)_j T(Z, Z_1) dZ_1}{\omega_0 \sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} \Gamma_0
\end{aligned}
$$

Equation (110) is a wave equation in an anisotropic medium, the anisotropy being described by the metric tensor $g_2^{ij}(Z)$.

8.6 Including transfer functions dynamics

Until now, we merely considered the dynamics of frequencies, and transfer functions were considered in first approximation as depending on frequencies. We will now briefly show how the model can be generalized by including dynamic oscillations for the transfer function. To simplify the formula, we consider a constant background frequency and restrain to the linear approximation defined by equation (99), but the computations can be generalized to a position dependent background (see appendix 7), and the idea can be translated to the non-local context presented in the next section.

To account for the dynamic nature of the transfer functions $T(Z, Z_1, \omega, \omega_1)$, we associate to equation (99) an evolution equation for $T(Z, Z_1, \omega, \omega_1)$. Using (343), we replace $T(Z, Z_1, \omega, \omega_1)$ by a general function $T(Z, Z_1, \theta)$ that is a priori independent from frequencies. Thus, around the equilibrium defined by the background frequency ω_0, the function $T(Z, Z_1, \theta)$ writes:

$$T(Z, Z_1, \theta) = T_0(Z, Z_1) + h(Z, Z_1)\hat{T}(Z, \theta, Z_1)$$

where $T_0(Z, Z_1)$ is the transfer function in this equilibrium. The function $\hat{T}(Z, \theta, Z_1)$ represents the fluctuations around this equilibrium. The expansion of G around ω_0 becomes:

$$\begin{aligned}
& G\left(\frac{\kappa}{N}\int \frac{\omega_0 + \Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) - \Omega(\theta, Z)}{\omega_0\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} T(Z, Z_1, \theta)\, dZ_1\right) \qquad (111)\\
\simeq\ & \omega_0 + \Gamma_0\left(\int\left(\frac{\Omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right) - \Omega(\theta, Z)}{\omega_0\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}} T_0(Z, Z_1) + \frac{h(Z, Z_1)\hat{T}(Z, \theta, Z_1)}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}}\right)\right)
\end{aligned}$$

As a consequence, equation (99) is replaced by:

$$\sigma_\theta^2\nabla_\theta^2\Omega(\theta, Z) = \Omega(\theta, Z) + \frac{\Gamma_1}{c}\nabla_\theta\Omega(\theta, Z) - \Gamma_2\nabla_Z^2\Omega(\theta, Z) - \frac{\Gamma_2}{c^2}\nabla_\theta^2\Omega(\theta, Z) - \Gamma_0\hat{T}(Z, \theta)$$

where we define:

$$\hat{T}(Z, \theta) = \int \frac{h(Z, Z_1)\hat{T}(Z, \theta, Z_1)}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}}$$

and:

$$\begin{aligned}
\Gamma_1 &= \frac{\kappa}{NX_r}\int \frac{|Z - Z_1|\, T_0(Z, Z_1)\, dZ_1}{\omega_0\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}}\Gamma_0 \\
\Gamma_2 &= \frac{\kappa}{2NX_r}\frac{\int (Z - Z_1)^2\, T_0(Z, Z_1)\, dZ_1}{\omega_0\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}}\Gamma_0 \\
\Gamma_0 &= G'\left(\frac{\kappa}{N}\int \frac{T_0(Z, Z_1)\, dZ_1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2 + \frac{\pi}{2}\alpha}}\right)
\end{aligned}$$

The dynamics for $\hat{T}(Z, \theta)$ derived in appendix 2 yields a system of dynamic equations for $\left(\Omega(\theta, Z), \hat{T}(Z, \theta)\right)$

that are similar to (99) for a slowly varying $\hat{T}(Z,\theta)$:

$$0 = \sigma_{\theta}^{2}\nabla_{\theta}^{2}\Omega(\theta,Z)-\left(\Omega(\theta,Z)+\frac{\Gamma_{1}}{c}\nabla_{\theta}\Omega(\theta,Z)-\Gamma_{2}\nabla_{Z}^{2}\Omega(\theta,Z)-\frac{\Gamma_{2}}{c^{2}}\nabla_{\theta}^{2}\Omega(\theta,Z)-\Gamma_{0}\hat{T}(Z,\theta)\right) \quad (112)$$

$$0 = \frac{\nabla_{\theta}^{2}\hat{T}(Z,\theta)}{\lambda}+U_{1}(\omega_{0})\nabla_{\theta}\hat{T}(Z,\theta)+U_{2}(\omega)\hat{T}(Z,\theta) \quad (113)$$
$$-\left(\rho\bar{C}(Z)h_{C}'(\omega_{0})-\frac{\rho\left(D(Z)\hat{T}_{0}(Z)h_{D}'(\omega_{0})+\bar{C}_{0}(Z)h_{C}'(\omega_{0})\right)}{\lambda\tau}\right)\Omega(Z,\theta)$$
$$-\frac{\rho D(Z)h_{D}'(\omega_{0})\left(\Gamma_{1}\nabla_{\theta}\Omega(Z,\theta)-\left(\Gamma_{1}\nabla_{\theta}^{2}\Omega(Z,\theta)+\Gamma_{2}\nabla_{Z}^{2}\Omega(Z,\theta)\right)\right)}{\lambda\tau}$$

with:

$$\bar{C}(Z) = \frac{1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_{r}}\right)^{2}+\frac{\pi}{2}\alpha}}\int h(Z,Z_{1})C(Z_{1})$$
$$\bar{C}_{0}(Z) = \frac{1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_{r}}\right)^{2}+\frac{\pi}{2}\alpha}}\int h(Z,Z_{1})C(Z_{1})\hat{T}_{0}(Z,Z_{1})$$
$$\hat{T}_{0}(Z) = \frac{1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_{r}}\right)^{2}+\frac{\pi}{2}\alpha}}\int h(Z,Z_{1})\hat{T}_{0}(Z,Z_{1})$$

Solving the system (112) and (113) implies, depending on the parameters, the existence of oscillatory solutions, both for frequencies of activity and transfer functions. These oscillatory solutions illustrate the constant interaction between cells' activity and the strength of connectivity between these cells.

To conclude this section, remark that in the limit of slowly varying transfer functions, equation (112) has constant coefficients, i.e. describes wave propagation in an homogeneous medium. However, beyond this approximation, equation (112) is replaced by:

$$\sigma_{\theta}^{2}\nabla_{\theta}^{2}\Omega(\theta,Z)=\Omega(\theta,Z)+\frac{\Gamma_{1}(\theta,Z)}{c}\nabla_{\theta}\Omega(\theta,Z)-\Gamma_{2}(\theta,Z)\nabla_{Z}^{2}\Omega(\theta,Z)-\frac{\Gamma_{2}(\theta,Z)}{c^{2}}\nabla_{\theta}^{2}\Omega(\theta,Z)$$

with:

$$\Gamma_{1}(\theta,Z) = \frac{\kappa}{NX_{r}}\int\frac{|Z-Z_{1}|T(Z,Z_{1},\theta)dZ_{1}}{\omega_{0}\sqrt{\frac{\pi}{8}\left(\frac{1}{X_{r}}\right)^{2}+\frac{\pi}{2}\alpha}}\Gamma_{0}$$
$$\Gamma_{2}(\theta,Z) = \frac{\kappa}{2NX_{r}}\frac{\int(Z-Z_{1})^{2}T(Z,Z_{1},\theta)dZ_{1}}{\omega_{0}\sqrt{\frac{\pi}{8}\left(\frac{1}{X_{r}}\right)^{2}+\frac{\pi}{2}\alpha}}\Gamma_{0}$$
$$\Gamma_{0}(\theta,Z) = G'\left(\frac{\kappa}{N}\int\frac{T(Z,Z_{1},\theta)dZ_{1}}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_{r}}\right)^{2}+\frac{\pi}{2}\alpha}}\right)$$

The dependency in (θ,Z) is driven by the oscillations (113) of the transfer functions. As a consequence, and as stated in section 8.5.2, the frequencies propagate as waves in an inhomogeneous medium, this inhomogeneity being time-dependent.

8.7 Some implications of the differential equation for frequencies in the linear approximation

To assess the implications of the wave equations and find the propagation of an external signal at some particular points to the whole thread, we must compute the Green functions associated to the linear approximation equations (99) and (109).

8.7.1 Green functions and external signals

The Green function of (97) and (109) are found using the usual Fourier representation. We focus on the retarded Green functions that model the wave propagation initiated by a source.

Constant background frequency We first consider (97). As explained in section 8.3, once the existence of stable solutions has been established, we can set:

$$\frac{\hat{f}_1}{\omega(\theta, Z)} + N_1 = 0$$

and replace (97) with its linear approximation (99) for $J(\theta) = 0$, that writes:

$$g_2 \nabla_Z^2 \Omega(\theta, Z) - \nabla_\theta^2 \Omega(\theta, Z) = g_0 \Omega(\theta, Z) \tag{114}$$

with:

$$g_0 = -\frac{f}{\left(\frac{\hat{f}_3}{\omega_0} - N_2\right)}, \; g_2 = -\frac{\frac{c^2 \hat{f}_3}{\omega_0}}{\left(\frac{\hat{f}_3}{\omega_0} - N_2\right)}$$

Given our assumptions in section 8.3, both g_0 and g_2 are positive.

Equation (114) is of Klein-Gordon type and can be normalized by setting $g_2 = 1$ and writing $g_0 = m^2$. Using its Fourier representation, the retarded Green function of (97) is given by:

$$\mathcal{G}(Z, Z', t, t') = \int dk \frac{\exp(ik.(Z - Z') - i\omega_k (t - t'))}{\omega_k} H(t - t') \tag{115}$$

with $\omega_k = \sqrt{k^2 + m^2}$. This integral can be computed and yields:

$$\mathcal{G}(Z, Z', t, t') = H(t - t') \left(\frac{1}{2\pi} \delta(t - t') - \frac{m J_1 \left(m \sqrt{(t - t')^2 - (Z - Z')^2} \right)}{\sqrt{(t - t')^2 - (Z - Z')^2}} \right) \tag{116}$$

where J_1 is the $n = 1$ Bessel function.

To inspect the implications of (116), we merely need to approximate it for small oscillations. For $g_0 >> g_2$, i.e. $m^2 > 1$, we can expand $\sqrt{k^2 + m^2}$ at the lowest order in $\frac{k^2}{m^2}$, and write (115), up to terms of order $\frac{1}{m^3}$, as:

$$\mathcal{G}(Z, Z', t, t') \simeq \int dk \frac{\exp\left(ik.(Z - Z') - i\left(m + \frac{k^2}{m}\right)(t - t')\right)}{m} H(t - t') \tag{117}$$

Computing the Fourier transform in (117), the function $\mathcal{G}_0(Z, Z', t, t')$ can be approximated by:

$$\mathcal{G}(Z, Z', t, t') = \exp\left(i \left(\frac{m}{2} \frac{(Z - Z')^2}{(t - t')} - m(t - t') \right) \right) H(t - t') \tag{118}$$

which shows that the Green function $\mathcal{G}(Z, Z', t, t')$ represents the path integral of a particle under the constant potential m.

Non-constant background frequency The Green function of equation (109) is a generalization of (116) and has been studied in the context of covariant quantum field theory. However, (118) shows that a path integral formulation for the Green function can produced. If $g_2(Z)$ varies slowly with Z, the analog of (118) with non-constant coefficients is:

$$\mathcal{G}(Z, Z', t, t') = \int \exp\left(i\left(\int_{z(t')=Z'}^{z(t)=Z}\left(\frac{\sqrt{\frac{g_0(z(s))}{g_2(z(s))}}}{2}\left(\frac{dz(s)}{ds}\right)^2 - \sqrt{g_0(z(s))}\right)ds\right)\right) Dz(s) H(t-t') \quad (119)$$

The sum is over the set of paths $z(s)$ starting from Z' and ending at Z in a time span of $t - t'$. The derivation of (119) is straightforward. If we neglect $g_1(Z)$ as in the derivation of (114), (109) writes:

$$\sigma_\theta^2 \nabla_\theta^2 \hat{\Omega}(\theta, Z) = g_0(Z)\hat{\Omega}(\theta, Z) + g_2(Z)\nabla_Z^2 \hat{\Omega}(\theta, Z)$$

We then cut the time span $t - t'$ into slices Δt, such that $g_0(Z)$ and $g_2(Z)$ can be considered constant in a domain of radius $c\Delta t$. The Green function for a time span Δt is given by a formula similar to (118), except that $g_2(Z) \neq 1$:

$$\mathcal{G}(z(s+\Delta t), z(s), \Delta t) = \exp\left(i\left(\frac{\sqrt{\frac{g_0(z(s))}{g_2(z(s))}}}{2}\frac{(z(s+\Delta t) - z(s))^2}{\Delta t} - g_0(z(s))\Delta t\right)\right) \quad (120)$$

Under these assumptions, the convolution of (120) over the time slices yields ultimately formula (119).

8.7.2 Propagation of external signals

Constant coefficients With the Green function (118), we can compute the diffusion of an external source along the thread by convolution. We assume an external source:

$$J(t, Z) = \exp(-i\omega_0 t)\,\delta(Z - Z_0) \quad (121)$$

which describes a signal located in Z_0, with frequency ω_0. Using (118), the amplitude $\Omega(t, Z)$ is:

$$\begin{aligned}\Omega(t, Z) &= \int \exp\left(i\left(\frac{m}{2}\frac{(Z - Z_0)^2}{(t-t')} - \omega_0 t - (m - \omega_0)(t - t')\right)\right) H(t-t')\,dt' \\ &= \frac{\exp\left(-i\omega_0 t - i\sqrt{m}\,|(m-\omega_0)|\,|Z - Z_0| + i\pi\right)}{\sqrt{|(m-\omega_0)|}}\end{aligned}$$

and for a signal including a whole range of frequencies:

$$\hat{f}(t, Z) = \int f(\omega_0)\exp(-i\omega_0 t)\,d\omega_0 \quad (122)$$

the corresponding response of the thread is:

$$\Omega(t, Z) = \int \frac{\exp\left(-i\omega_0 t - i\sqrt{m}\,|(m-\omega_0)|\,|Z - Z_0| + i\pi\right)}{\sqrt{|(m-\omega_0)|}} f(\omega_0)\,d\omega_0$$

We assume that the range of frequencies in (122) is such that $m - \omega_0 > 0$, so that:

$$\begin{aligned}\Omega(t, Z) &= \int \frac{\exp\left(-i\omega_0 t - i\sqrt{m}\,(m-\omega_0)\,|Z - Z_0| + i\pi\right)}{\sqrt{|(m-\omega_0)|}} f(\omega_0)\,d\omega_0 \\ &= \int \frac{\exp\left(-i\omega_0\left(t - \sqrt{m}\,|Z - Z_0|\right)\right)}{\sqrt{|(m-\omega_0)|}} f(\omega_0)\,d\omega_0 \frac{\exp\left(-i\left(\sqrt{m}\right)^3 |Z - Z_0| + i\pi\right)}{\sqrt{|(m-\omega_0)|}}\end{aligned}$$

To simplify, we also assume that the frequencies of the signal satisfy $|\omega_0| << m$, so that:

$$\Omega(t,Z) \simeq \hat{f}\left(t-\sqrt{m}\,|Z-Z_0|\,,Z_0\right)\frac{\exp\left(-i\left(\sqrt{m}\right)^3|Z-Z_0|+i\pi\right)}{m} \tag{123}$$

The whole past history of the signal is present in the frequencies at time t, and is thus recorded in the system of oscillations. The result (123) can be extended for several independent sources. When these sources are located in two points Z_1, Z_2 that emit signals $\hat{f}_1(t)$ and $\hat{f}_2(t)$ respectively, with frequencies below m, the response is:

$$\begin{aligned}\Omega(t,Z) \quad \simeq \quad & \hat{f}_1\left(t-\sqrt{m}\,|Z-Z_0|\right)\frac{\exp\left(-i\left(\sqrt{m}\right)^3|Z-Z_0|+i\pi\right)}{\sqrt{|(m-\omega_0)|}} \\ & +\hat{f}_2\left(t-\sqrt{m}\,|Z-Z_0|\right)\frac{\exp\left(-i\left(\sqrt{m}\right)^3|Z-Z_0|+i\pi\right)}{\sqrt{|(m-\omega_0)|}}\end{aligned} \tag{124}$$

The response defined by (124) may present some interference phenomena, depending on $\hat{f}_1$ and $\hat{f}_2$, as usual in waves dynamics.

Non constant coefficients Considering non constant coefficients in (119) translates the hypothesis of position-dependent transfer functions between cells. The implications of this assumption may be understood using formula (124). Assume a thread divided in two regions, each characterized by constant coefficients g_0 and g_2 and only connected via two "entry points". This can be modelled by $g_2 = 0$ on the border of the two regions, and $g_2 >> 1$ at these two points.

Formula (119) implies that paths that do not cross the border at points Z_1or Z_2 do not contribute to the Green function. Actually, the factor $\sqrt{\frac{g_0(z(s))}{g_2(z(s))}}$ that arises in the weight (119) of such paths induces large oscillations in the vicinity of the border that cancel the contribution of the paths.

As a consequence, the paths contributing to the Green function have to cross at Z_1or Z_2, which induces some interference phenomena (124) on the transmitted signal.

More generally, the dependency of the transfer functions in Z along the paths impacts the results, even for simple signals (121). Actually, the various paths reaching a point Z of the thread contribute to the Green function (124). They each acquire a phase that depends on both the path and the characteristic of the medium encountered. These phases may create interferences between the paths. The trained networks may present some particular learned features in the coefficients $g_0(Z)$ and $g_2(Z)$, i.e. their transfer functions, that would produce either constructive or destructive interferences for the signals.

Non-static equilibrium The equations of the previous paragraph may be generalized for a non-constant and slowly varying background solution. For a non-static potential and for currents of large magnitude, a slowly varying solution of the type:

$$\begin{aligned}\omega_0(\theta,Z) \quad = \quad & F\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega_0\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega_0(\theta,Z)}\right. \\ & \left.\times W\left(\frac{\omega_0(\theta,Z)}{\omega_0\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\mathcal{G}_0(0,Z_1)+\left|\Psi_0\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

may exist, and we can expand (85) around $\omega_0(\theta,Z)$ in series of $\delta\Psi$. Minimizing the effective action (see appendix 3) yields the values of $\delta\Psi$. Equation (95) and the definition of the coefficients (94) are still valid, but now $\bar{\mathcal{G}}_0(0,Z_1) = \mathcal{G}_0(0,Z_1)+|\Psi_0|^2$ has to be replaced by a time dependent propagator $\bar{\mathcal{G}}_0(\theta,Z_1) = \mathcal{G}_0(0,Z_1)+\left|\Psi_0\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2$. The coefficients arising in (95) thus become time dependent.

9 Beyond local approximation

9.1 One-field system

Section 8 focused on the wave equation for frequencies, i.e. the local solutions of (85), plus the corrections defined in (84). This section goes one step further and studies the dynamics for frequencies without the locality assumption. To do so, we dismiss the correction terms due to the interaction between the system and the stabilization potential in (84), and thus study the solutions of (85) by rewriting the equation:

$$\begin{aligned}\omega(J,\theta,Z) &= F\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)W\left(\frac{\omega(\theta,Z)}{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)}{\omega(\theta,Z)}\right.\\&\left.\times\left(\bar{\mathcal{G}}_0(0,Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}\tag{125}$$

9.1.1 Series expansion of (125)

To write a non-local solution of (125), we use the series expansion in $\left|\Psi\left(\theta^{(j)},Z_1\right)\right|^2$ of the right-hand side of (306) and write:

$$\begin{aligned}\omega(J,\theta,Z) &= \omega(\theta,Z)_{|\Psi|^2=0}\\&+\int\sum_{n=1}^{\infty}\left(\frac{\delta^n\omega(J,\theta,Z)}{\prod\limits_{i=1}^{n}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\right)_{|\Psi|^2=0}\prod_{i=1}^{n}\left|\Psi(\theta-l_i,Z_i)\right|^2\end{aligned}\tag{126}$$

The first term in (126), $\omega\left(\theta^{(i)},Z\right)_{|\Psi|^2=0}$, is a solution of:

$$\omega(\theta,Z)_{|\Psi|^2=0}=F\left(J+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega_{|\Psi|^2=0}\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega_{|\Psi|^2=0}(\theta,Z)}W\left(\frac{\omega_{|\Psi|^2=0}(\theta,Z)}{\omega_{|\Psi|^2=0}\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\bar{\mathcal{G}}_0(0,Z_1)\right)dZ_1\right)\tag{127}$$

To find the internal dynamics of the system, we will first consider a constant external current $J(\theta)=J$, typically $J=0$, but the results of this section will be valid for a non static current $J(\theta)$. The static solution of (127) satifies:

$$\begin{aligned}\omega(J,Z) &= F\left(J+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega(Z_1)}{\omega(Z)}W\left(\frac{\omega(Z)}{\omega(Z_1)}\right)\bar{\mathcal{G}}_0(0,Z_i)\,dZ_1\right)\\&\equiv F[J,\omega,Z]\end{aligned}$$

we assume this solution to be known, and we chose to expand $\omega(J,\theta,Z)$ in (126) around this solution, the dynamics being given by $\left|\Psi\left(\theta^{(j)},Z_1\right)\right|^2$. We thus set:

$$\omega(\theta,Z)_{|\Psi|^2=0}=\omega(J,Z)$$

Appendices 5 and 6 compute the derivatives $\left(\frac{\delta^n\omega(J,\theta,Z)}{\prod_{i=1}^{n}\delta|\Psi(\theta-l_i,Z_i)|^2}\right)_{|\Psi|^2=0}$ in (126).

Defining:

$$\begin{aligned}&\hat{T}(\theta,Z,Z_1,\omega,\Psi)\\=&\frac{\frac{\kappa}{N}\omega(J,\theta,Z)\,T(Z,Z_1)\,F'[J,\omega,\theta,Z,\Psi]}{\omega^2(J,\theta,Z)+\left(\int\frac{\kappa}{N}\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\left(\bar{\mathcal{G}}_0(0,Z')+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\right)T(Z,Z')\,dZ'\right)F'[J,\omega,\theta,Z,\Psi]}\end{aligned}\tag{128}$$

and the operator $\hat{T}$ with kernel:

$$\hat{T}\left(\left(Z^{(l-1)},\theta^{(l-1)}\right),\left(Z^{(l)},\theta^{(l)}\right)\right) = \hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0\right) \times\delta\left(\left(\theta^{(l)}-\theta^{(l-1)}\right)-\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right) \tag{129}$$

appendix 5 shows that:

$$\frac{\delta\omega\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2} = \sum_{n=1}^{\infty}\int\frac{\omega\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)}{\left(\bar{\mathcal{G}}_0\left(0,Z_1\right)+\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2\right)} \times\prod_{l=1}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)} \tag{130}$$

Appendix 6 builds on (130) to compute the derivative arising in the series expansion (126):

$$\left(\frac{\delta^n\omega\left(J,\theta,Z\right)}{\prod_{i=1}^{n}\delta\left|\Psi\left(\theta-l_i,Z_i\right)\right|^2}\right)_{|\Psi|^2=0}\prod_{i=1}^{n}\left|\Psi\left(\theta-l_i,Z_i\right)\right|^2 \tag{131}$$

by a graphical representation. We associate the squared field $\left|\Psi\left(\theta-l_i,Z_i\right)\right|^2$ to each point Z_i and draw m lines for $m=1,...,n$. One of them at least is starting from Z. These lines are composed of an arbitrary number of segments and all the points Z_i are crossed by one line. Each line ends at a point Z_i. The starting points of the lines branch either at Z or at some point of an other line. There are m branching points of valence k including the starting point at Z. Apart from Z, the branching points have valence $3,...,n-1$.

To each line i of length L_i, we associate the factor:

$$\begin{aligned} F\left(line_i\right) &= \prod_{l=1}^{L_i}\frac{\frac{\kappa}{N}T\left(Z^{(l-1)},Z^{(l)}\right)F'\left[J,\omega_0,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)}\right]}{\omega_0\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)}\right)} \\ &\quad\times\frac{\omega_0\left(J,\theta-\sum_{l=1}^{L_i}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_i\right)}{\bar{\mathcal{G}}_0\left(0,Z_i\right)} \\ &= \prod_{l=1}^{L_i}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0,\Psi\right)\frac{\omega_0\left(J,\theta-\sum_{l=1}^{L_i}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_i\right)}{\bar{\mathcal{G}}_0\left(0,Z_i\right)} \end{aligned} \tag{132}$$

and to each branching point $(X,\theta)=B$ of valence $k+2$, we associate the factor:

$$F\left((X,\theta)\right)=\frac{\delta^k\left(\frac{\frac{\kappa}{N}T\left(Z,Z^{(l)}\right)F'\left[J,\theta,\omega_0,Z^{(l)}\right]\bar{\mathcal{G}}_0\left(0,Z^{(l)}\right)}{\omega_0\left(J,\theta,Z^{(l)}\right)}\right)}{\delta^k\omega_0\left(J,\theta,Z^{(l)}\right)} \tag{133}$$

and (131) writes as a series of lines contributions connected by the branching points:

$$\left(\frac{\delta^n \omega(J,\theta,Z)}{\prod_{i=1}^{n} \delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\right)_{|\Psi|^2=0} \prod_{i=1}^{n}\left|\Psi(\theta-l_i,Z_i)\right|^2 \tag{134}$$

$$= \left(\sum_{m=1}^{n}\sum_{i=1}^{m}\sum_{(line_1,\dots,line_m)}\prod_i F(line_i)\prod_B F(B)\right)\prod_{i=1}^{n}\left|\Psi(\theta-l_i,Z_i)\right|^2$$

The graphical representation is generic. The integration over the set of lines also accounts for the degenerate case of lines that share some segments.

9.1.2 Path integral description

Formalism Appendix 6 uses formula (134) to derive a non-local formula for the successive derivatives of $\omega(J,\theta,Z)$ and $\omega^{-1}(J,\theta,Z)$. Moreover, equation (134) allows to rewrite the expansion (307) as the sum of graphs for an auxiliary complex field $\Lambda(Z_i,\theta_i)$. The idea is to regroup the graphs in (134) so that their sum becomes a sum over graphs drawn between an arbitrary number of branch points, seen as vertices of arbitrary valence k and associated factor (133). These vertices are connected by the edges of the graph with associated Green functions $\frac{1}{1-\left(1+|\Psi|^2\right)\hat{T}}$ where $\hat{T}$ is the operator whose kernel is defined in (129). The factor $|\Psi|^2$ is the operator multiplication by $|\Psi(\theta,Z)|^2$ at point (θ,Z).

Appendix 6 shows that:

$$\begin{aligned}\omega(\theta,Z) &= \omega_0(J,\theta,Z)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\int\hat{T}\Lambda^{\dagger}(Z,\theta)\int\prod_{i=1}^{n}\omega_0(J,\theta_i,Z_i)\left|\Psi(J,\theta_i,Z_i)\right|^2\Lambda(Z_i,\theta_i)\,d(Z_i,\theta_i)\exp(-S(\Lambda))\mathcal{D}\Lambda}{\exp(-S(\Lambda))\mathcal{D}\Lambda}\\ &= \omega_0(J,\theta,Z)+\frac{\int\hat{T}\Lambda^{\dagger}(Z,\theta)\exp\left(-S(\Lambda)+\int\Lambda(X,\theta)\omega_0(J,\theta,Z)\left|\Psi(J,\theta,Z)\right|^2 d(X,\theta)\right)\mathcal{D}\Lambda}{\int\exp(-S(\Lambda))\mathcal{D}\Lambda}\end{aligned} \tag{135}$$

The action for the fields Λ and $\Lambda^{\dagger}$ is:

$$\begin{aligned}S(\Lambda) &= \int\Lambda(Z,\theta)\left(1-|\Psi|^2\hat{T}\right)\Lambda^{\dagger}(Z,\theta)\,d(Z,\theta)\\ &\quad-\int\Lambda(Z,\theta)\hat{T}\left(\theta-\frac{\left|Z-Z^{(1)}\right|}{c},Z,Z^{(1)},\omega_0+\hat{T}\Lambda^{\dagger}\right)\Lambda^{\dagger}\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta\end{aligned}$$

with:

$$\begin{aligned}&\hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_0+\hat{T}\Lambda^{\dagger}\right)\\ &= \hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_0(Z,\theta)+\int\hat{T}\left(\theta-\frac{\left|Z-Z^{(1)}\right|}{c},Z^{(1)},Z,\omega_0\right)\Lambda^{\dagger}\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZ^{(1)}\right)\end{aligned}$$

We then show that, in the saddle point approximation, the detrended frequency:

$$\Omega(\theta,Z)=\omega(\theta,Z)-\omega_0(J,\theta,Z)$$

satisfies the following equation:

$$\Omega-\hat{T}(\Omega+\omega_0)|\Psi|^2-\hat{T}\frac{\omega_0\Omega}{\omega_0+\Omega}=0 \tag{136}$$

Equation (136) can be used in two ways.

Series expansion A first application of the frequencies equation (136) considers the background field as an external field $|\Psi|^2$. This case arises when the system is coupled to an external source $J(Z,\theta)$ that shapes the background field. A solution of (136) can be found as a series expansion in the field $|\Psi|^2$ (see appendix 6). The dominant terms of the series is:

$$\begin{aligned}
\omega(Z,\theta) &= \omega_0(J,\theta,Z) + \int \sum_{k=0}^{\infty} \frac{\exp\left(-c\sum_{i=0}^{k} l_i - \alpha\left(1+\left\langle|\Psi|^2\right\rangle\right)\left(\sum_{i=0}^{k}(cl_i)^2 - \sum_{l=0}^{k-1}\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{B^{k+1}} \\
&\times \prod_{i=1}^{k}\left(\frac{\omega_0(\theta-l_i,Z_i)}{\omega_0(\theta-l_i,Z_i)+A\omega_0|\Psi|^2(\theta-l_i,Z_i)}\right)\frac{\omega_0(J,\theta-l_k,Z_k)}{\left(1+\left\langle|\Psi|^2\right\rangle\right)}|\Psi(\theta-l_k,Z_k)|^2 dZ_i dl_i \quad (137)
\end{aligned}$$

where $\left\langle|\Psi|^2\right\rangle$ is the average of $|\Psi|^2$ over the thread.

For $\omega^{-1}(Z,\theta)$, we obtain:

$$\begin{aligned}
\omega^{-1}(Z,\theta) &= \omega_0^{-1}(J,\theta,Z) \\
&+\frac{G'[J,\omega,\theta,Z,\Psi]}{F'[J,\omega,\theta,Z,\Psi]}\int\sum_{k=0}^{\infty}\frac{\exp\left(-c\sum_{i=0}^{k} l_i - \alpha\left(1+\left\langle|\Psi|^2\right\rangle\right)\left(\sum_{i=0}^{k}(cl_i)^2 - \sum_{l=0}^{k-1}\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{D^{k+1}} \\
&\times \prod_{i=1}^{k}\left(\frac{\omega_0(\theta-l_i,Z_i)}{\omega_0(\theta-l_i,Z_i)+A\omega_0|\Psi|^2(\theta-l_i,Z_i)}\right)\frac{\omega_0^{-1}(J,\theta-l_k,Z_k)}{\left(1+\left\langle|\Psi|^2\right\rangle\right)}|\Psi(\theta-l_k,Z_k)|^2 dZ_i dl_i \quad (138)
\end{aligned}$$

The full series expansion for $\omega(Z,\theta)$ is derived in appendix 6.

Non local differential equation Equation (136) can also be used to obtain a non-local version of the frequencies equation (85). To do so, we replace the background field by a function of the frequencies:

$$\Psi(\theta,Z) = \frac{\nabla_\theta \omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}{U''(X_0)\,\omega^2\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}\Psi_0(\theta,Z)$$

that rewrites as:

$$|\Psi|^2 = \frac{X_0^2}{U''(X_0)}\frac{\nabla_\theta\Omega}{(\omega_0+\Omega)^2}$$

Equation (136) thus becomes:

$$\Omega - \hat{T}\frac{\frac{X_0^2}{U''(X_0)}\nabla_\theta\Omega+\omega_0\Omega}{\omega_0+\Omega} = 0 \quad (139)$$

This is a non-linear equation that generalizes (85). The second-order expansion in derivatives of the right-hand side of (139) yields a second order linear differential equation similar to the type derived in sections 8.2 and 8.3 and confirms the possibility of waves propagation phenomenom.

Remark that equation (139) is still valid for any background field related to ω by a relation of the type:

$$|\Psi|^2 = f\left(\omega,\nabla_\theta^l\omega\right) \quad (140)$$

which leads to the following dynamic equation:

$$\Omega - \hat{T}\omega f\left(\omega,\nabla_\theta^l\omega\right) - \hat{T}\frac{\omega_0(\omega-\omega_0)}{\omega} = 0$$

Corrections to the saddle point Ultimately, appendix 6 also yields the corrections to the saddle point approximation. Replacing equation (135) by:

$$\omega=\omega_0+\hat{T}\Lambda_0^{\dagger}\left(\det\left(1+\left(1+\frac{\omega_0\left(2\omega_0+\hat{T}\Lambda_0^{\dagger}\right)}{\left(\omega_0+\hat{T}\Lambda_0^{\dagger}\right)^2}\hat{T}\left(1+|\Psi|^2\right)\hat{T}\right)^{-1}\right)\right)^{-1} \tag{141}$$

and considering equation (141) along with the defining equation for $\Lambda_0^{\dagger}$:

$$\Lambda_0^{\dagger}-\left(|\Psi|^2+\frac{\omega_0}{\omega_0+\hat{T}\Lambda_0^{\dagger}}\right)\hat{T}\Lambda_0^{\dagger}-\omega_0|\Psi|^2=0 \tag{142a}$$

yields the modified version of (139):

$$\begin{aligned}
&\Omega\det\left(1+\left(1+\frac{\omega_0\left(2\omega_0+\Omega\right)}{\left(\omega_0+\Omega\right)^2}\hat{T}\left(1+f\left(\omega,\nabla_\theta^l\omega\right)\right)\hat{T}\right)^{-1}\right)\\
=\;&\hat{T}\left(\left(f\left(\omega,\nabla_\theta^l\omega\right)+\frac{\omega_0}{\omega_0+\Omega\det\left(1+\left(1+\frac{\omega_0(2\omega_0+\Omega)}{(\omega_0+\Omega)^2}\hat{T}\left(1+f\left(\omega,\nabla_\theta^l\omega\right)\right)\hat{T}\right)^{-1}\right)}\right)\right.\\
&\left.\times\Omega\det\left(1+\left(1+\frac{\omega_0\left(2\omega_0+\Omega\right)}{\left(\omega_0+\Omega\right)^2}\hat{T}\left(1+f\left(\omega,\nabla_\theta^l\omega\right)\right)\hat{T}\right)^{-1}\right)+\omega_0 f\left(\omega,\nabla_\theta^l\omega\right)\right)
\end{aligned}$$

9.2 Several interacting fields

The results of section 9.1 can be extended in the case of two types of interactions. Consider n populations, each caracterized by their frequencies $i=1,...,n$, and interacting either positively or negatively. Each population is defined by a field Ψ_i and frequencies $\omega_i(\theta,Z)$. Equations for frequencies are defined by (43):

$$\begin{aligned}
\omega_i(\theta,Z)\;=\;&F_i\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega_i(\theta,Z)}G^{ij}\right.\\
&\left.\times W\left(\frac{\omega_i(\theta,Z)}{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\bar{\mathcal{G}}_{0j}(0,Z_1)+\left|\Psi_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)
\end{aligned} \tag{143}$$

The coefficients of the $n\times n$ matrix G belong to the interval $[-1,1]$. The sum over indices is implicit for j.

The resolution of (143) is similar to that of (85), but with a vector of frequencies. The series expansion of this vector is:

$$\begin{aligned}
\omega(\theta,Z)\;=\;&\omega(\theta,Z)_{|\Psi|^2=0}\\
&+\int\sum_{n=1}^{\infty}\left(\frac{\delta^n\omega(J,\theta,Z)}{\prod_{i=1}^{n}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\right)_{|\Psi|^2=0}\prod_{i=1}^{n}\left|\Psi(\theta-l_i,Z_i)\right|^2
\end{aligned} \tag{144}$$

where the expression of the first order derivative is similar to (130):

$$\begin{aligned}
\left(\frac{\delta\omega(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2}\right)_{|\Psi|^2=0}\;=\;&\sum_{n=1}^{\infty}\int\prod_{l=1}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0,0\right)\\
&\times\Omega_0\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)\times\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}
\end{aligned}$$

where $\omega\left(\theta^{(i)},Z\right)$ and $\omega\left(\theta^{(i)},Z\right)_{|\Psi|^2=0}=\omega_0$ are vectors of frequencies. We define $D\left(|\Psi|^2\right)$ as a diagonal matrix with components $|\Psi_i|^2$ on the diagonal. More generally, for any expression $H\left(\omega_{0i},|\Psi_i|^2\right)$, we define $D\left(H\left(\omega_0,|\Psi|^2\right)\right)$ as the diagonal matrix with components $H\left(\omega_{0i},|\Psi_i|^2\right)$. The expressions $\left(\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\right)_{|\Psi|^2=0}$ and $\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0,0\right)$ are $n\times n$ matrices:

$$\left(\left(\frac{\delta\omega\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2}\right)_{|\Psi|^2=0}\right)_{ij}=\left(\frac{\delta\omega_i\left(J,\theta,Z\right)}{\delta\left|\Psi_j\left(\theta-l_1,Z_1\right)\right|^2}\right)_{|\Psi|^2=0}$$

$$\begin{aligned}&\hat{T}_{ij}\left(\theta,Z,Z_1\omega,\Psi\right)\\ =&\frac{G^{ij}\frac{\kappa}{N}\omega_i\left(J,\theta,Z\right)T\left(Z,Z_1\right)F'\left[J,\omega,\theta,Z,\Psi\right]}{\omega_i^2\left(J,\theta,Z\right)+G^{ij}\left(\int\frac{\kappa}{N}\omega_j\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\left(\bar{\mathcal{G}}_{0j}\left(0,Z'\right)+\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\right)T\left(Z,Z'\right)dZ'\right)F'\left[J,\omega,\theta,Z,\Psi\right]}\end{aligned}$$

and the operator $\hat{T}$ with kernel:

$$\begin{aligned}\hat{T}\left(\left(Z^{(l-1)},\theta^{(l-1)}\right),\left(Z^{(l)},\theta^{(l)}\right)\right)&=\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0\right)\\&\times\delta\left(\left(\theta^{(l)}-\theta^{(l-1)}\right)-\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\end{aligned}$$

The successives derivatives in (144) are given by a formula similar to (134)

$$\left(\frac{\delta^n\omega\left(J,\theta,Z\right)}{\prod_{i=1}^{n}\delta\left|\Psi\left(\theta-l_i,Z_i\right)\right|^2}\right)_{|\Psi|^2=0}=\left(\sum_{m=1}^{n}\sum_{i=1}^{m}\sum_{(line_1,...,line_m)}\prod_i F\left(line_i\right)\prod_B F\left(B\right)\right)\tag{145}$$

where the various quantities $\left(\frac{\delta^n\omega(J,\theta,Z)}{\prod_{i=1}^{n}\delta|\Psi(\theta-l_i,Z_i)|^2}\right)_{|\Psi|^2=0}$, $F\left(line_i\right)$ and $F\left(B\right)$ are tensors whose precise form and dimensions are given in appendix 6.3.

The resolution for frequencies follows the single field case, and yields (see appendix 6.3):

$$\Omega-\hat{T}\left(\Omega+\omega_0\right)|\Psi|^2-\hat{T}\left(\frac{\omega_0}{\omega_0+\Omega}\right)\Omega=0$$

where $\left(\Omega+\omega_0\right)|\Psi|^2$ and $\left(\frac{\omega_0}{\omega_0+\Omega}\right)\Omega$ are the vectors with components $\left(\Omega+\omega_0\right)_i|\Psi|_i^2$ and $\left(\frac{\omega_{i0}}{\omega_{i0}+\Omega_i}\right)\Omega_i$, respectively. The approximate series expansion for $\omega\left(Z,\theta\right)$ and $\omega^{-1}\left(Z,\theta\right)$ are given in appendix 6.3:

$$\begin{aligned}\omega\left(Z,\theta\right)&=\omega_0\left(J,\theta,Z\right)+\int\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}\frac{\exp\left(-cl_i-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left(\left(cl_i\right)^2-\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{B}\qquad(146)\\&\times D\left(\frac{\omega_0\left(\theta-l_i,Z_i\right)}{\omega_0\left(\theta-l_i,Z_i\right)+A\omega_0\left|\Psi\right|^2\left(\theta-l_i,Z_i\right)}\frac{\omega_0\left(J,\theta-l_k,Z_k\right)}{\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)}\right)\\&\times\frac{\exp\left(-cl_k-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left(\left(cl_i\right)^2-\frac{|Z_{k-1}-Z_k|}{c}\right)\right)}{B}\left|\Psi\left(\theta-l_k,Z_k\right)\right|^2dZ_idl_i\qquad(147)\end{aligned}$$

$$
\begin{aligned}
\omega^{-1}(Z,\theta) &= \omega_0^{-1}(J,\theta,Z) \\
&\quad + D\left(\frac{G'[J,\omega,\theta,Z,\Psi]}{F'[J,\omega,\theta,Z,\Psi]}\right)\int\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}\frac{\exp\left(-cl_i-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_i)^2-\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{B} \\
&\quad \times D\left(\frac{\omega_0(\theta-l_i,Z_i)}{\omega_0(\theta-l_i,Z_i)+A\omega_0|\Psi|^2(\theta-l_i,Z_i)}\frac{\omega_0(J,\theta-l_k,Z_k)}{\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)}\right) \\
&\quad \times\frac{\exp\left(-cl_k-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_i)^2-\frac{|Z_{k-1}-Z_k|}{c}\right)\right)}{B}|\Psi(\theta-l_k,Z_k)|^2\,dZ_i dl_i
\end{aligned}
\tag{148}
$$

The full series expansion for $\omega(Z,\theta)$ is given in the same appendix.

10 Correlation functions and probabilistic interpretation

The correlation functions of the field theory can be interpreted in terms of the system's dynamics. They compute the joint probability for a set of frequencies at different points during a certain interval of time. We first compute and interpret the two points correlation functions. We then generalize to an arbitrary number of points. It is in this context that the interdependence of frequencies at different points appear.

10.1 Two points correlation functions

The correlation functions are found by computing the derivatives of the effective action with respect to the classical background field. The two points Green function is the inverse of the second derivative of the effective action $\Gamma\left(\Psi^{\dagger},\Psi\right)$:

$$
\Gamma_{1,1}\left((\theta_f,Z_f),(\theta_i,Z_i)\right)=\frac{\delta^2\Gamma\left(\Psi^{\dagger},\Psi\right)}{\delta\Psi^{\dagger}(\theta_f,Z_f)\,\delta\Psi(\theta_i,Z_i)}
$$

In first approximation, we have:

$$
\Gamma_{1,1}\left((\theta_f,Z_f),(\theta_i,Z_i)\right)=-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\delta(\theta_f-\theta_i)+\hat{\Gamma}_{1,1}\left((\theta_f,Z_f),(\theta_i,Z_i),\Psi^{\dagger},\Psi\right)
\tag{149}
$$

with $\hat{\Gamma}_{1,1}\left((\theta_f,Z_f),(\theta_i,Z_i),\Psi^{\dagger},\Psi\right)$ given by the second derivative of (57):

$$
\begin{aligned}
&\hat{\Gamma}_{1,1}\left((\theta_f,Z_f),(\theta_i,Z_i),\Psi^{\dagger},\Psi\right) \\
&= \sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}\sum_{l'_f=1,l'_i=1}^{j}\int\frac{\left(\prod_{l=1,l\neq l'_f}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)}{m!\prod_k(\sharp_k)!}\times\frac{\prod_{l=1}^{j}\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{\Lambda^{\sum_{i,l}p_l^i}} \\
&\quad\times\prod_{i=1}^{m}\left[\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right]_{\substack{\left(\theta^{\left(l'_f\right)},Z_{l'_f}\right)=(\theta_f,Z_f)\\ \left(\theta^{\left(l'_i\right)},Z_{l'_i}\right)=(\theta_i,Z_i)}}\times\left(\prod_{l=1,l\neq l'_i}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)
\end{aligned}
\tag{150}
$$

In these expressions, the derivatives that correspond to the impact of propagation between θ_i and θ_f of the signal have been neglected. In the local approximation, equation (150) writes:

$$\hat{\Gamma}_{1,1}\left(\left(\theta,Z_f\right),\left(\theta,Z_i\right),\Psi^{\dagger},\Psi\right) \tag{151}$$

$$= \sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}\sum_{l_f'=1,l_i'=1}^{j}\int\left[\left(\prod_{l=1,l\neq l_f'}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\right.$$

$$\left.\times\frac{\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right]}{m!\prod_k\left(\sharp_k\right)!\Lambda_1^j\Lambda^{\sum_{i,l}p_l^i}}\left(\prod_{l=1,l\neq l_i'}^{j}\Psi\left(\theta^{(l)},Z_l\right)d\theta^{(l)}dZ_l\right)\right]_{\substack{\left(\theta^{(l')},Z_{l_f'}\right)=\left(\theta,Z_f\right)\\ \left(\theta^{(l')},Z_{l_i'}\right)=\left(\theta,Z_i\right)}} \tag{152}$$

with $p_l^i=0$ or 1. The two points correlation function is then:

$$G_2\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right)\right)=\mathcal{G}\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right)\right)+\mathcal{G}*\sum_{n\geqslant 2}\left(-1\right)^{n-1}\left(\hat{\Gamma}_{1,1}\left(\Psi^{\dagger},\Psi\right)*\mathcal{G}\right)^n \tag{153}$$

where $\mathcal{G}\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right)\right)$ satisfies:

$$-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\right)\mathcal{G}\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right)\right)=\delta\left(\theta_f-\theta_i\right)$$

and $\hat{\Gamma}_{1,1}\left(\Psi^{\dagger},\Psi\right)$ is the operator with kernel $\hat{\Gamma}_{1,1}\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right),\Psi^{\dagger},\Psi\right)$. Appendix 4.1 yields an expression for $\hat{\Gamma}_{1,1}\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right),\Psi^{\dagger},\Psi\right)$ and $\mathcal{G}\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right)\right)$ in the approximation of relatively slow variations of frequencies. We find:

$$\hat{\Gamma}_{1,1}\left(\left(\theta,Z_f\right),\left(\theta,Z_i\right),\Psi^{\dagger},\Psi\right)=\omega^{-1}\left(\theta_i,Z_i\right)\omega^{-1}\left(\theta_f,Z_f\right)\Psi\left(\theta_f,Z_f\right)\Psi^{\dagger}\left(\theta_i,Z_i\right)C\left(\bar{\omega},\Psi\right)$$

where:

$$C\left(\bar{\omega},\Psi\right)=\frac{1}{2D\Lambda}\frac{1}{2D\Lambda}\exp\left(\int\frac{\omega^{-1}\left(\bar{\theta},\bar{Z}\right)}{\Lambda_1}d\bar{\theta}d\bar{Z}\right)\times\exp\left(-\int\frac{c}{2D\Lambda}\left|\Psi\left(\bar{\theta},\bar{Z}\right)\right|^2d\bar{\theta}d\bar{Z}\right)$$

Moreover, we define the average frequency at Z over a time span $[\theta,\theta']$ as:

$$\bar{\omega}^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\equiv\left\langle\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\right\rangle_{[\theta,\theta']}$$

so that we obtain:

$$\mathcal{G}\left(\theta_f,\theta_i,Z_f,Z_i\right) \tag{154}$$

$$\simeq\delta\left(Z_f-Z_i\right)\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{\exp\left(-\left(\sqrt{\left(\frac{\bar{\omega}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\sigma^2}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{\bar{\omega}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\sigma^2}\right)\left(\theta-\theta'\right)\right)}{\sqrt{\left(\frac{\bar{\omega}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\sigma^2}\right)^2+\frac{2\alpha}{\sigma^2}}}H\left(\theta-\theta'\right)$$

As a consequence, the solution of (153) is the series expansion:

$$
\begin{aligned}
G_2(\theta_f,\theta_i,Z_f,Z_i) \simeq\;& \mathcal{G}(\theta_f,\theta_i,Z_f,Z_i) \\
&+\int\left(\prod_{k=1}^{n} d\theta_k dZ_k\right)\mathcal{G}(\theta_f,\theta_n,Z_f,Z_f)\frac{\Psi(\theta_n,Z_f)}{\omega(\theta_n,Z_f)}\times\sum_{n\geqslant 2}(-1)^{n-1} \\
&\times\prod_{k=1}^{n-1}\left(\left|\Psi(\theta_k,Z_k)\right|^2\left(\omega^{-1}(\theta_k,Z_k)\right)^2\mathcal{G}(\theta_{k+1},\theta_k,Z_k,Z_k)\right)^{n-1}\frac{\Psi^{\dagger}(\theta_1,Z_i)}{\omega(\theta_1,Z_i)}\mathcal{G}(\theta_1,\theta_i,Z_i,Z_i)
\end{aligned}
\tag{155}
$$

In the following, we study the correlation functions with an arbitrary number of points.

10.2 (k,n) points correlation functions

The (k,n) points correlation functions are derived in the standard way (see appendix 4.2).

The correlation functions are obtained from the (k,n)-th effective vertex $\Gamma_{k,n}$:

$$
\Gamma_{k,n}\left(\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k},\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}\right)=\frac{\delta^{k+n}\Gamma\left(\Psi^{\dagger},\Psi\right)}{\delta^k\left(\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)_{l=1,..,k}\delta^n\left(\Psi\left(\theta_i^{(l)},Z_l\right)\right)_{l=1,..,n}}
$$

through standard techniques. Appendix 4 shows that, in first approximation:

$$
\begin{aligned}
& G_{k,n}\left(\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k},\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}\right) \\
=\;& \sum_{\sigma_k,\sigma_n}\sum_{u=0}^{\inf(k,n)}\prod_{j=0}^{u}G_2\left(\left(\theta_f^{(j)},Z_f\right),\left(\theta_i^{(i)},Z_i\right)\right) \\
&\times\sum_{i=1,j=1}^{k-u,n-u}(-1)^{i+j}\sum_{\substack{P_i(k-u)\\P_j(n-u)}}\prod_{\substack{r\in P_i(k-u)\\s\in P_j(n-u)}}\Gamma_{k_r,n_s}\left(\left(\theta_f^{(l,r,u)},Z_{l,r,u}\right)_{l=1,..,k_r},\left(\theta_i^{(l,s,u)},Z_{l,s,u}\right)_{l=1,..,n_s}\right)
\end{aligned}
\tag{156}
$$

where $P_i(k)$ and $P_j(n)$ denote the partitions of k and n in i and j subsets:

$$
\cup_r\left(\theta_f^{(l,r,u)},Z_{l,r,u}\right)_{l=1,..,k_r}=\left(\theta_f^{(l)},Z_l\right)_{l=u+1,..,k}
$$

and:

$$
\cup_s\left(\theta_i^{(l,s,u)},Z_{l,s,u}\right)_{l=u+1,..,n_s}=\left(\theta_i^{(l)},Z_l\right)_{l=u+1,..,n}
$$

as ordered sets. The sum over σ_k and σ_n is over all permutations of the $\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k}$ and $\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}$, respectively.

10.3 Interpretation: joined probabilities for frequencies

Equations (155) and (156) can be interpreted in terms of joined probabilities for frequencies at different points of the thread. We first consider the two points correlation functions.

10.3.1 Two points functions

At the perturbative zeroth order, the function $\mathcal{G}_0(\theta,\theta',Z)$ is the Green function of the operator:

$$
-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0\right)\right)+\alpha
$$

and is given by (47):

$$\mathcal{G}_0\left(\theta,\theta',Z,Z'\right)=\delta\left(Z-Z'\right)\frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{1}{\sigma^2\bar{X}_r}\right)\left(\theta-\theta'\right)\right)}{\Lambda}H\left(\theta-\theta'\right)$$

This function is the Laplace transform of the function $\hat{G}_{0Z}\left(\theta,\theta',\Delta n\right)$:

$$\mathcal{G}_0\left(\theta,\theta',Z,Z'\right)=\int\hat{G}_{0Z}\left(\theta,\theta',\Delta n\right)\exp\left(-\alpha\Delta n\right)d\alpha$$

The form of $\hat{G}_{0Z}\left(\theta,\theta',\Delta n\right)$ is irrelevant here.

The function $\hat{G}_{0Z}$ computes the probability of a time interval $\theta-\theta'$ for Δn spikes of the potential at point Z. The Laplace transform $\mathcal{G}_0\left(\theta,\theta',Z,Z'\right)$ computes the probability of a time interval $\theta-\theta'$ for a random number of spikes Δn with average $\frac{1}{\alpha}$. Since the spikes' frequency is $\frac{\Delta n}{\theta-\theta'}$, $\mathcal{G}_0\left(\theta,\theta',Z,Z'\right)$ computes the average probability of a frequency $\frac{1}{\alpha(\theta-\theta')}$ of spikes. Computing the average $\langle(\theta-\theta')\rangle$ confirms this point:

$$\begin{aligned}\mathcal{G}_0\left(\theta,\theta',Z,Z'\right)&=\delta\left(Z-Z'\right)\frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{1}{\sigma^2\bar{X}_r}\right)\left(\theta-\theta'\right)\right)}{\Lambda}H\left(\theta-\theta'\right)\\&\simeq\delta\left(Z-Z'\right)\frac{\exp\left(-\alpha\bar{X}_r\left(\theta-\theta'\right)\right)}{\Lambda}H\left(\theta-\theta'\right)\end{aligned}$$

so that $\langle(\theta-\theta')\rangle=\frac{1}{\alpha\bar{X}_r}$. The average inverse frequency is then $\alpha\langle(\theta-\theta')\rangle=\frac{1}{\bar{X}_r}$.

As a consequence, the Green function $\mathcal{G}_0\left(\theta,\theta',Z,Z'\right)$ computed at $\alpha=1$ can be interpreted as the probability, at time $\frac{\theta+\theta'}{2}$ and position Z, of a spikes' frequency equal to $\frac{1}{\theta-\theta'}$. The same applies for higher order correlation functions.

Including the higher order corrections (154) and (155) lead to the same conclusion, but using (155) for $Z_f=Z_i$ shows the interdependencies of frequencies. Actually, this can be rewritten:

$$\begin{aligned}G_2\left(\theta_f,\theta_i,Z_i,Z_i\right)&\simeq\mathcal{G}\left(\theta_f,\theta_i,Z_i,Z_i\right)\\&+\int\left(\prod_{k=1}^{n}d\theta_k dZ_k\right)\mathcal{G}\left(\theta_f,\theta_n,Z_i,Z_i\right)\frac{\Psi\left(\theta_n,Z_f\right)}{\omega\left(\theta_n,Z_f\right)}\times\sum_{n\geqslant2}\left(-1\right)^{n-1}\\&\times\prod_{k=1}^{n-1}\left(\frac{\Psi^{\dagger}\left(\theta_k,Z_k\right)}{\omega^{-1}\left(\theta_k,Z_k\right)}\mathcal{G}\left(\theta_{k+1},\theta_k,Z_k,Z_k\right)\frac{\Psi\left(\theta_k,Z_k\right)}{\omega\left(\theta_k,Z_k\right)}\right)\frac{\Psi^{\dagger}\left(\theta_1,Z_i\right)}{\omega\left(\theta_1,Z_i\right)}\mathcal{G}\left(\theta_1,\theta_i,Z_i,Z_i\right)\end{aligned}\tag{157}$$

The probability $\mathcal{G}\left(\theta_f,\theta_i,Z_i,Z_i\right)$ of frequency $\theta_f-\theta_i$ on an interval centered on $\frac{\theta_f+\theta_i}{2}$ is modified recursively by probabilities $\mathcal{G}\left(\theta_{k+1},\theta_k,Z_k,Z_k\right)$ at other points and times with a factor $\frac{\Psi(\theta_k,Z_k)}{\omega(\theta_k,Z_k)}\frac{\Psi^{\dagger}(\theta_1,Z_i)}{\omega(\theta_1,Z_i)}$. This factor measures the rates of interaction between different points. The probability $\mathcal{G}\left(\theta_1,\theta_i,Z_i,Z_i\right)$ impacts $\mathcal{G}\left(\theta_2,\theta_1,Z_1,Z_1\right)$, that itself impacts $\mathcal{G}\left(\theta_3,\theta_2,Z_2,Z_2\right)$ and so on, until $\mathcal{G}\left(\theta_f,\theta_n,Z_i,Z_i\right)$ closes the series of successive modifications. The sum over times and space yields the impact of the whole system on the frequencies at Z_i.

10.3.2 (n,n) Green functions

The (n,n) Green function $G_{n,n}\left(\left(\theta_f^{(l)},Z_l\right)_{l=1,..,n},\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}\right)$ computes the transition probability of $\left(\theta_i^{(n)},Z_l\right)_{l=1,..,n}$ to $\left(\theta_f^{(n)},Z_l\right)_{l=1,..,n}$ for $i=1...l$ for an average number of spikes of $\frac{1}{\alpha}$, so that:

$$\begin{aligned}&\left(G_{n,n}\left(\left(\theta_f^{(n)},Z_l\right)_{l=1,..,n},\left(\theta_i^{(n)},Z_l\right)_{l=1,..,n}\right)\right)_{\alpha=1}\\&=P\left(\omega\left(Z_1,\frac{\theta_f^{(1)}+\theta_i^{(1)}}{2}\right)=\frac{1}{\theta_f^{(1)}-\theta_i^{(1)}},...,\omega\left(Z_l,\frac{\theta_f^{(n)}+\theta_i^{(n)}}{2}\right)=\frac{1}{\theta_f^{(n)}-\theta_i^{(n)}}\right)\end{aligned}\tag{158}$$

computes the joined probability for a set of n frequencies at points Z_1,..., Z_n and times $\frac{\theta_f^{(1)}+\theta_i^{(1)}}{2}$,..., $\frac{\theta_f^{(n)}+\theta_i^{(n)}}{2}$. Equation (158) can be rewritten in terms of density for the set of variables $\theta^{(l)}=\frac{\theta_f^{(l)}+\theta_i^{(l)}}{2}$ and $\omega^{(i)}=\frac{1}{\theta_f^{(l)}-\theta_i^{(l)}}$:

$$P\left(\left(\omega^{(i)},\theta^{(i)}\right)_{1\leqslant i\leqslant n}\right)=\left(\prod_{l=1}^{n}\frac{1}{\left(\omega^{(l)}\right)^2}\right)\left(G_{n,n}\left(\left(\theta_f^{(n)},Z_l\right)_{l=1,..,n},\left(\theta_i^{(n)},Z_l\right)_{l=1,..,n}\right)\right)_{\alpha=1}$$

Using (158), we can now interpret equations (155) and (156). Writing:

$$\begin{aligned}&G_{n,n}\left(\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k},\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}\right)\\=&\sum_{\sigma_n,\sigma_n}\sum_{u=0}^{n}\prod_{j=0}^{u}G_2\left(\left(\theta_f^{(j)},Z_f\right),\left(\theta_i^{(i)},Z_i\right)\right)\\&\times\sum_{i=1,j=1}^{n-u}(-1)^{i+j}\sum_{\substack{P_i(n-u)\\P_j(n-u)}}\prod_{\substack{r\in P_i(n-u)\\s\in P_j(n-u)}}\Gamma_{n_r,n_s}\left(\left(\theta_f^{(l,r,u)},Z_{l,r,u}\right)_{l=1,..,n_r},\left(\theta_i^{(l,s,u)},Z_{l,s,u}\right)_{l=1,..,n_s}\right)\end{aligned}$$

in terms of probabilities:

$$\begin{aligned}&\left(\prod_{l=1}^{n}\left(\omega^{(l)}\right)^2\right)P\left(\left(\omega^{(i)},\theta^{(i)}\right)_{1\leqslant i\leqslant n}\right)\\=&\sum_{\sigma_n,\sigma_n}\sum_{u=0}^{n}\prod_{j=0}^{u}\left(\omega^{(j)}\right)^2P\left(\omega^{(j)},\theta^{(j)}\right)\sum_{i=1,j=1}^{n-u}(-1)^{i+j}\\&\times\sum_{\substack{P_i(n-u),\\P_j(n-u)}}\prod_{\substack{r\in P_i(n-u)\\s\in P_j(n-u)}}\Gamma_{n_r,n_s}\left(\left(\theta^{(l,r,u)}+\frac{\left(\omega^{(l,r,u)}\right)^{-1}}{2},Z_{l,r,u}\right)_{l=1,..,n_r},\left(\theta^{(l,r,u)}-\frac{\left(\omega^{(l,r,u)}\right)^{-1}}{2},Z_{l,s,u}\right)_{l=1,..,n_s}\right)\end{aligned}$$

has an interpretation similar to the 2-points Green function. The first terms $\prod_{j=0}^{u}\left(\omega^{(j)}\right)^2P\left(\omega^{(j)},\theta^{(j)}\right)$ represent an independent distribution for the frequencies at different points, and the corrective terms measure the mutual dependencies due to the interactions in the background field. Moreover, for $l=m=1$, the probabilistic interpretation is an alternate description to the frequencies' local differential equation.

11 Conclusion

We have presented a field theoretic framework for a system with a large number of interacting spiking neurons, and showed its implications on the dynamics of the system frequencies.

The field framework and the existence of collective or background states allow for stable traveling wave solutions and correlated frequencies at different points. These correlations are measured by the n point Green functions and induces a non-locality in frequencies wave equations. which we accounted for by deriving non local equations for the frequencies. Besides, some non-locality also emerges in the impact of the external current on the background field. An external current may shape the form of the background field, which in turn conditions the thread in which frequencies waves propagate.

We have presented several further extensions of our framework. First, we have extended our formalism to multi-component fields, to include different types of cells interacting with each others. Second, we have accounted for the possibility of time and position-dependent transfer functions, where the dependency results from the strength of cells interactions. This extension induces frequencies' wave equations with non constant coefficients, that are waves in a non-homogeneous medium, whose non-homogeneity is described by a metric.

Our results have been obtained using a dynamic evolution for transfer functions that depends on the background field for the system of cells. A straightforward extension would be to design a field formalism for the transfer functions themselves, in interaction with the cells' field.

Appendix 0

When we restrict the fields to those of the form:

$$\Psi(\theta, Z)\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right) \tag{159}$$

where $\omega^{-1}(J,\theta,Z,\Psi)$ satisfies:

$$\begin{aligned}
&\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right) \\
=\;& G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1\right)
\end{aligned} \tag{160}$$

The classical effective action writes:

$$-\frac{1}{2}\int\Psi^{\dagger}(\theta,Z)\,\Psi(\theta,Z)\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\left(\left(\frac{\sigma^2}{2}\nabla_\theta-\omega^{-1}\right)\nabla_\theta\right)\Psi(\theta,Z)\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right) \tag{161a}$$

We can replace the first δ function by 1 to normalize the projection on the frequency dependent states.. The action of ∇_θ on $\Psi(\theta,Z)\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)$ yields:

$$\begin{aligned}
&\nabla_\theta\left(\Psi(\theta,Z)\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\right) \\
=\;&(\nabla_\theta\Psi(\theta,Z))\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right) \\
&-\left(\nabla_\theta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\Psi(\theta,Z)\,\delta'\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)
\end{aligned} \tag{162}$$

Inserting the result (162) in (161a) leads to:

$$\begin{aligned}
&-\frac{1}{2}\int\Psi^{\dagger}(\theta,Z)\,\Psi(\theta,Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta-\omega^{-1}\right)\right)(\nabla_\theta\Psi(\theta,Z))\,\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right) \\
&+\frac{1}{2}\int\Psi^{\dagger}(\theta,Z)\,\Psi(\theta,Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta-\omega^{-1}\right)\right)\Psi(\theta,Z)\,\delta'\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right) \\
=\;&-\frac{1}{2}\int\Psi^{\dagger}(\theta,Z)\,\Psi(\theta,Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\right)\nabla_\theta\Psi(\theta,Z) \\
&-\frac{1}{2}\int\Psi^{\dagger}(\theta,Z)\,\Psi(\theta,Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta-\nabla_\theta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\right)\Psi(\theta,Z)
\end{aligned}$$

and the sum of the two last terms is, as in the text:

$$-\frac{1}{2}\int\Psi^{\dagger}(\theta,Z)\left(\nabla_\theta\left(\frac{\sigma^2}{2}\nabla_\theta-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\right)\Psi(\theta,Z)$$

Appendix 1. Vertices of (35) involved in the computation of the $2n$ Green functions

To find the effective action associated to (35) and the collective term (34), we proceed in several steps. The first one is to find the vertices involved in the computation of the Green functions. To do so, we will expand the action (35) in series of field. This produces a series of an infinite series of vertices. However, given that the two points Green function are not symmetric by time reversal, we will show that only the $2n$ first terms are involved in the computation of the $2n$ Green functions. We will then estimate these vertices using the recursive relation (27) between frequencies depending on field. These results will be used in the next section to find the graph expansion of the system's partition function.

1.1 Estimation of the two points Green function

We start with the two points Green function and prove (50). To do so, we will expand the action functional in series of the field Ψ. The two points Green functions will be computed by using the "free" action's propagator, obtained by replacing $\omega^{-1}(J,\theta,Z,\Psi)$ with $\omega^{-1}(J,\theta,Z,0)$ in (35). The free action is:

$$S_0=-\frac{1}{2}\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\left(\frac{\sigma^2}{2}\nabla_{\theta}-\omega^{-1}(J,\theta,Z,0)\right)\Psi(\theta,Z) \tag{163}$$

and the series in field of (35) will be considered, as usual, as a perturbation expansion.

1.1.1 "Free" action propagator.

Now, we compute the propagator associated to (163). We decompose the external current into a static and a time dependent parts $\bar{J}+J(\theta)$ where $\bar{J}$ can be thought as the time average of the current. We will consider that $\left|\bar{J}(Z)\right|>|J(\theta,Z)|$. At zeroth order in current $J(\theta)$, the function $\omega^{-1}(J,\theta,Z,0)$ satisfies:

$$\begin{aligned}\omega^{-1}(J,\theta,Z,0) &= G\left(\bar{J}+J(\theta)\right) \qquad (164)\\ &\simeq G\left(\bar{J}(Z)\right)=\frac{\arctan\left(\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{\bar{J}(Z)}\right)}{\sqrt{\bar{J}(Z)}}=\frac{1}{\bar{X}_r(Z)}\equiv\frac{1}{\bar{X}_r}\end{aligned}$$

where the dependence in Z of $\bar{X}_r$ will be understood. As a consequence $\omega(\theta,Z)$ is thus approximatively equal to $\bar{X}_r$. Under this approximation:

$$S_0=-\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\left(\frac{\sigma^2}{2}\nabla_{\theta}-\frac{1}{\bar{X}_r}\right)\Psi(\theta,Z)$$

and the Green function of the operator $\nabla_{\theta}\left(\frac{\sigma^2}{2}\nabla_{\theta}-\frac{1}{\bar{X}_r}\right)$ is computed as:

$$\left\langle\Psi^{\dagger}(\theta,Z)\Psi(\theta',Z)\right\rangle\equiv\mathcal{G}_0((\theta,Z),(\theta',Z'))\equiv\mathcal{G}_0(\theta,\theta',Z)=\delta(Z-Z')\int\frac{\exp(ik(\theta-\theta'))}{\frac{\sigma^2}{2}k^2+ik\frac{1}{\bar{X}_r}+\alpha}dk \tag{165}$$

The right hand side of (165) can be computed as:

$$\begin{aligned}\int\frac{\exp(ik(\theta-\theta'))}{\frac{\sigma^2}{2}k^2+ik\frac{1}{\bar{X}_r}+\alpha}dk &= \exp\left(\frac{\theta-\theta'}{\sigma^2\bar{X}_r}\right)\int\frac{\exp(ik(\theta-\theta'))}{\frac{\sigma^2}{2}k^2+\frac{1}{2}\left(\frac{1}{\sigma\bar{X}_r}\right)^2+\alpha}dk\\ &= \frac{1}{\sqrt{\frac{\pi}{2}}}\frac{\exp\left(-\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}\left|\theta-\theta'\right|\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}}\exp\left(\frac{\theta-\theta'}{\sigma^2\bar{X}_r}\right) \qquad (166)\end{aligned}$$

and this is quickly suppressed for $\theta-\theta'<0$. This is the direct consequence of non-hermiticity of operator. In the sequel, for $\sigma^2\bar{X}_r<<1$, we can thus consider that:

$$\mathcal{G}_0(\theta,\theta',Z)=\delta(Z-Z')\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{1}{\sigma^2\bar{X}_r}\right)(\theta-\theta')\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}}H(\theta-\theta') \tag{167}$$

where H is the Heaviside function:

$$\begin{aligned}H(\theta-\theta') &= 0 \text{ for } \theta-\theta'<0\\ &= 1 \text{ for } \theta-\theta'>0\end{aligned}$$

Formula (167) for the propagator is sufficient to compute the graphs expansion in the next paragraphs. We can check that the corrections due to a non-static current do not modify the result at a good level of approximation. Considering the following form for $G(J(\theta, Z))$:

$$G(J(\theta, Z)) = \frac{\arctan\left(\left(\frac{1}{X_r} - \frac{1}{X_p}\right)\sqrt{J(\theta, Z)}\right)}{\sqrt{J(\theta, Z)}}$$

For relatively high frequency firing rates, i.e., small periods of time between two spikes, we can write in first approximation:

$$\begin{aligned} G\left(\bar{J} + J(\theta, Z)\right) &\simeq G\left(\bar{J}\right) + J(\theta, Z)\, G'\left(\bar{J}\right) \\ &= \frac{1}{\bar{X}_r} + J(\theta, Z)\, G'\left(\bar{J}\right) \end{aligned}$$

and replace (165) by the Green function of:

$$\nabla_\theta\left(\frac{\sigma^2}{2}\nabla_\theta - G(J(\theta, Z))\right) \simeq \nabla_\theta\left(\frac{\sigma^2}{2}\nabla_\theta - \frac{1}{\bar{X}_r} - J(\theta, Z)\, G'\left(\bar{J}\right)\right)$$

As a consequence, the inverse frequency $\mathcal{G}_0(\theta, \theta', Z)$ defined in (167) is replaced by:

$$\begin{aligned} \mathcal{G}_0\left((\theta, Z), (\theta', Z')\right) &= \delta(Z - Z')\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}} - \frac{1}{\sigma^2\bar{X}_r}\right)(\theta - \theta')\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}} H(\theta - \theta') \\ &\quad\times\left(1 - \frac{1}{\sqrt{\frac{\pi}{2}}}\frac{G'\left(\bar{J}\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}}\int_\theta^{\theta'} J(\theta'', Z)\, d\theta''\right) \end{aligned}$$

Since $J(\theta, Z)$ is a deviation around the static part $\bar{J}$, the corrective term:

$$-\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{G'\left(\bar{J}\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2 + \frac{2\alpha}{\sigma^2}}}\int_\theta^{\theta'} J(\theta'', Z)\, d\theta''$$

vanishes quickly as $\theta - \theta'$ increases, which justifies approximation (167).

1.1.2 perturbation expansion and the two points Green function

Formula (167) allows to compute higher order contributions to the Green function of action (35) by using a graph expansion. Actually, writing $\omega^{-1}(\theta, Z)$ for $\omega^{-1}(J, \theta, Z, \Psi)$ when no ambiguity is possible, the higher order contribution for the series expansion of $\omega^{-1}(\theta, Z)$ in fields are obtained by solving recursively:

$$\omega^{-1}(J, \theta, Z) = G\left(J(\theta, Z) + \int \frac{\kappa}{N}\frac{\omega\left(J, \theta - \frac{|Z - Z_1|}{c}, Z_1\right)}{\omega(J, \theta, Z)}\left|\Psi\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)\right|^2 T(Z, \theta, Z_1)\, dZ_1 d\omega_1\right) \tag{168}$$

This will be done precisely in the next paragraph. For now, it is enough to note that given (168), the recursive expansion in $\omega^{-1}(J, \theta, Z)$ of the potential term in (35):

$$\frac{1}{2}\Psi^\dagger(\theta, Z)\nabla\left(G\left(J(\theta, Z) + \int \frac{\kappa}{N}\frac{\omega\left(J, \theta - \frac{|Z - Z_1|}{c}, Z_1\right)}{\omega(J, \theta, Z)}\left|\Psi\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)\right|^2 T(Z, Z_1)\, dZ_1\right)\right)\Psi(\theta, Z) \tag{169}$$

induces the presence of products in the series expansion of the two points Green function:

$$\prod_{i=1}^{m}\int\Psi^{\dagger}\left(\theta^{(i)},Z_{i}\right)\nabla_{\theta^{(i)}}\prod_{k=1}^{k_{i}}\left(\prod_{l=1}^{l_{k}}\prod_{\alpha(l)=1}^{n(\alpha(l))}\int\left|\Psi\left(\theta^{(i)}-\frac{\left|Z_{i}-Z_{\alpha(l)}^{(1)}\right|+...+\left|Z_{\alpha(l)}^{(l-1)}-Z_{\alpha(l)}^{(l)}\right|}{c},Z_{\alpha(l)}^{(l)}\right)\right|^{2}\right)$$
$$\times dZ_{\alpha(l)}^{(1)}...dZ_{\alpha(l)}^{(l_{k})}\Psi\left(\theta^{(i)},Z_{i}\right)d\theta^{(i)}dZ_{i} \tag{170}$$

with $n(\alpha(l))\geqslant n(\alpha(l'))$ for $l>l'$ and $m\in\mathbb{N}$. The function $\delta(Z-Z')$ in (165) and the use of Wick's theorem imply that all subgraphs drawn from this product reduce to a product of free Green functions (167) of the following form (the gradient terms and the indices $\alpha(l)$ are not included and do not impact the reasoning):

$$\int\prod_{i}\mathcal{G}_{0}\left(\theta^{(i)}-\sum_{l\leqslant n_{i}}\frac{\left|Z_{i}-Z_{i}^{(l)}\right|}{c},\theta^{(i+1)}-\sum_{k\leqslant n_{i+1}}\frac{\left|Z_{i+1}-Z_{i+1}^{(k)}\right|}{c},Z_{i}^{(n_{i})},Z_{i}^{(n_{i+1})}\right)$$
$$\times\delta\left(Z_{1}-Z_{i}^{(n_{i})}\right)\delta\left(Z_{1}-Z_{i+1}^{(n_{i+1})}\right)dZ_{i}^{(n_{i})}dZ_{i+1}^{(n_{i+1})}\prod_{i}d\theta^{(i)}$$
$$=\int\prod_{i}\mathcal{G}_{0}\left(\theta^{(i)}-\sum_{l\leqslant n}\frac{\left|Z_{i}-Z_{1}^{(l)}\right|}{c},\theta^{(i+1)}-\sum_{k\leqslant m}\frac{\left|Z_{i+1}-Z_{1}^{(k)}\right|}{c},Z_{1}\right)\prod_{i}d\theta^{(i)}$$
$$=\int\prod_{i}\mathcal{G}_{0}\left(\theta^{(i)},\theta^{(i+1)},Z_{1}\right)\prod_{i}d\theta^{(i)} \tag{171}$$

by change of variable in the successive integrations. Moreover, the cancelation of $\mathcal{G}_0(\theta,\theta',Z)$ for $\theta<\theta'$ implies that this product is different from zero only for $\theta^{(i)}<\theta^{(i+1)}$. As a consequence, for all closed loops $\theta_1<...<\theta^{(i)}<\theta^{(i+1)}<...\theta_n=\theta_1$, the contribution (171) for loop graphs reduces to:

$$\prod_{i}\mathcal{G}_{0}(\theta_{1},\theta_{1},Z_{1})=\prod_{i}\mathcal{G}_{0}(0,Z_{1})$$

with (see (167)):

$$\mathcal{G}_{0}(0,Z)=\frac{1}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^{2}X_{r}}\right)^{2}+\frac{2\pi\alpha}{\sigma^{2}}}}$$

As a consequence, the contribution of (170) to the two points Green function between an initial and final state:

$$\left\langle\Psi^{\dagger}(\theta_{in},Z_{in})\int\prod_{i=1}^{m}\Psi^{\dagger}\left(\theta^{(i)},Z_{i}\right)\right.$$
$$\times\nabla_{\theta^{(i)}}\prod_{k=1}^{k_{i}}\left(\left(\prod_{l=1}^{l_{k}}\int\left|\Psi\left(\theta^{(i)}-\frac{\left|Z_{i}-Z^{(1)}\right|+...+\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z^{(l)}\right)\right|^{2}dZ^{(1)}...dZ^{(l_{k})}\right)\right)$$
$$\left.\times\,\Psi\left(\theta^{(i)},Z_{i}\right)d\theta^{(i)}dZ_{i}\Psi(\theta_{fn},Z_{fn})\right\rangle \tag{172}$$

reduces to sums and integrals of the type:

$$\delta(Z_{in}-Z_{fn})\sum_{p}\mathcal{G}_{0}(\theta_{in},\theta_{1},Z_{in})\mathcal{G}_{0}(\theta_{1},\theta_{2},Z_{in})...\mathcal{G}_{0}(\theta_{p},\theta_{fn},Z_{in}) \tag{173}$$
$$\times\left(\sum_{n}\sum_{\left\{L_{1}^{(p)},...,L_{n}^{(p)}\right\}}\prod_{m=1}^{n}\left(\mathcal{G}_{0}(0,0,Z_{m})\right)^{l\left(L_{m}^{(p)}\right)}\right)$$

where $\left\{L_1^{(p)},...,L_n^{(p)}\right\}$ is the set of all n-uplet of possible closed loops that can be drawn from the remaining variables in (172) once p variables have been chosen.

The result (173) is the same as if in (169) the potential had been expanded to the second order in Ψ and in all terms of higher order, $|\Psi(\theta,Z)|^2$ had been replaced by $\mathcal{G}_0(0,Z)$.

Now, writing $\omega\left(J,\theta,Z,|\Psi|^2\right)$ for ω and $\omega(0)=\omega(J,\theta,Z,0)$ (i.e. when we set $\Psi\equiv 0$), this means that the 2 points Green functions are computed using the free action:

$$\begin{aligned}
&-\frac{1}{2}\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}(0)\right)\Psi(\theta,Z) \qquad (174)\\
&+\frac{1}{2}\Psi^{\dagger}(\theta,Z)\sum_{n>0}\frac{\nabla_{\theta}\left(\omega^{-1}\right)^{([n])}(0)}{[n]!}\left(\mathcal{G}_0(0,Z)\right)^{n}\Psi(\theta,Z)\\
&+\sum_{n>0}\left(\nabla_{\theta}\frac{\left(\omega^{-1}\right)^{([n-1])}(0)|\Psi|^{2}}{[n-1]!}\left(\mathcal{G}_0(0,Z)\right)^{n-1}\mathcal{G}_0(\theta,\theta',Z)\right)_{\theta'=\theta}\\
=\;&-\frac{1}{2}\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}(0)\right)\Psi(\theta,Z)+\frac{1}{2}\Psi^{\dagger}(\theta,Z)\sum_{n>0}\nabla_{\theta}\left(\left(\omega^{-1}\right)\left(\mathcal{G}_0(0,Z)\right)-\omega^{-1}(0)\right)\Psi(\theta,Z)\\
&+\Psi^{\dagger}(\theta,Z)\left(\nabla_{\theta'}\left(\left(\omega^{-1}\right)^{([1])}\left(\mathcal{G}_0(0,Z)\right)\Psi(\theta',Z)\mathcal{G}_0(\theta,\theta',Z)\right)\right)_{\theta'=\theta}\\
\equiv\;&-\frac{1}{2}\Psi^{\dagger}(\theta,Z)\left(\nabla_{\theta}\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}\right)\Psi(\theta,Z)+\frac{1}{2}|\Psi|^{2}\left[\frac{\delta\left[\Psi^{\dagger}(\theta',Z)\nabla_{\theta}\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\Psi(\theta,Z)\right]}{\delta|\Psi|^{2}}\right]_{|\Psi(\theta,Z)|^{2}=\mathcal{G}_0(0,Z)}
\end{aligned}$$

where $\frac{\left(\omega^{-1}\right)^{([n])}(0)}{[n]!}$ is a short notation for:

$$\sum_{l_i}\int\prod_{i=1}^{n}dZ_{l_i}^{(1)}...dZ_{l_i}^{(l_i)}\left(\frac{\delta^{n}\left[\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\right]}{\prod_{i=1}^{n}\delta\left(\left|\Psi\left(\theta-\frac{\left|Z-Z_{l_i}^{(1)}\right|+...+\left|Z^{l_i(l-1)}-Z_{l_i}^{(l_i)}\right|}{c},Z_{l_i}^{(l_i)}\right)\right|^{2}\right)}\right)_{|\Psi|=0}$$

and $\frac{\left(\omega^{-1}\right)^{([n-1])}(0)|\Psi|^{2}}{[n-1]!}$ stands for:

$$\begin{aligned}
&\sum_{l_i}\int\prod_{i=1}^{n-1}dZ_{l_i}^{(1)}...dZ_{l_i}^{(l_i)}\left(\frac{\delta^{n-1}\left[\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\right]}{\prod_{i}\delta\left(\left|\Psi\left(\theta-\frac{\left|Z-Z_{l_i}^{(1)}\right|+...+\left|Z_{l_i}^{(l-1)}-Z_{l_i}^{(l_i)}\right|}{c},Z_{l_i}^{(l_i)}\right)\right|^{2}\right)^{k_{l_i}}}\right)_{|\Psi|=0}\\
&\times\sum_{j=1}^{n-1}\left|\Psi\left(\theta-\frac{\left|Z-Z_{l_j}^{(1)}\right|+...+\left|Z_{l_i}^{(l-1)}-Z_{l_j}^{(l_j)}\right|}{c},Z_{l_j}^{(l_j)}\right)\right|^{2}
\end{aligned}$$

Similar notation is valid for $\frac{\left(\omega^{-1}\right)^{([n])}\left(\mathcal{G}_0(0,0,Z)\right)|\Psi|^{2}}{[n-1]!}$, the derivatives are evaluated at $|\Psi(\theta,Z)|^{2}=\mathcal{G}_0(0,0,Z)$.

We have also used $|\Psi|^2 \left[\frac{\delta}{\delta|\Psi|^2}\right]$ as a shorthand for:

$$\sum_l \int \left(\frac{dZ_l^{(1)}...dZ_l^{(l)}}{(k_l)!}\right) \left|\Psi\left(\theta - \frac{\left|Z - Z_l^{(1)}\right| + ... + \left|Z_l^{(l-1)} - Z_l^{(l_j)}\right|}{c}, Z_l^{(l)}\right)\right|^2 \tag{175}$$
$$\times \frac{\delta}{\delta\left(\left|\Psi\left(\theta - \frac{\left|Z-Z_l^{(1)}\right|+...+\left|Z_l^{(l-1)}-Z_l^{(l)}\right|}{c}, Z_l^{(l)}\right)\right|^2\right)}$$

Ultimately, the computation of the Green function involves the series expansion of the potential $V(\Psi)$. We have seen above (see equation(173)) that the graphs generated by this expansion are the same as if in (169) the potential had been expanded to the second order in Ψ and in all terms of higher order, $|\Psi(\theta, Z)|^2$ had been replaced by $\mathcal{G}_0(0, Z)$. As a consequence, the second order Green functions are computed with the action:

$$-\frac{1}{2}\Psi^\dagger(\theta, Z)\left(\nabla_\theta \frac{\sigma_\theta^2}{2}\nabla_\theta\right)\Psi(\theta, Z)$$
$$+\frac{1}{2}|\Psi|^2 \left[\frac{\delta\left[\Psi^\dagger(\theta', Z)\nabla_\theta\left(\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\Psi(\theta, Z)\right)\right]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}} + |\Psi|^2\left[\frac{\delta[V(\Psi)]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}$$

Equivalently, this means that the 2 points Green functions are the inverse of the operator:

$$-\frac{1}{2}\nabla_\theta \frac{\sigma_\theta^2}{2}\nabla_\theta + \frac{1}{2}\left[\frac{\delta\left[\Psi^\dagger(\theta', Z)\nabla_\theta\left(\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\Psi(\theta, Z)\right)\right]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}} + \left[\frac{\delta[V(\Psi)]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}$$

1.2 Higher order vertices involved in the effective action

1.2.1 General form of the vertices

To compute the $2n$ points Green functions, we proceed as for the two points function and consider a series expansion of the potential in powers of $\Psi(\theta, Z)$. In products $\prod_{i=1}^{n}|\Psi(\theta_i, Z_i)|^2$, $n-k$ factors $|\Psi(\theta_i, Z_i)|^2$ are replaced by $\mathcal{G}_0(0, 0, Z_i)$ at the higher orders. A derivation similar to (174) then shows that $2n$ Green functions are computed by using the expansion of the action:

$$-\frac{1}{2}\Psi^\dagger(\theta, Z)\left(\nabla_\theta \frac{\sigma_\theta^2}{2}\nabla_\theta\right)\Psi(\theta, Z) \tag{176}$$
$$+\frac{1}{2}\sum_{n\geqslant k\geqslant 0}|\Psi|^{2k}\left(\frac{\delta^k}{[k]!\delta^k|\Psi|^2}\left[\Psi^\dagger(\theta', Z)\nabla_\theta\left(\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\Psi(\theta, Z)\right)\right]\right)_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}$$

where $|\Psi|^{2k}\ \frac{\delta^k}{[n]!\delta^k|\Psi|^2}$ generalizes (175) and stands for:

$$\sum_{l_i}\int\prod_{i=1}^{k}\left(dZ_{l_i}^{(1)}...dZ_{l_i}^{(l_i)}\right)\left|\Psi\left(\theta - \frac{\left|Z - Z_{l_j}^{(1)}\right| + ... + \left|Z_{l_i}^{(l-1)} - Z_{l_j}^{(l_j)}\right|}{c}, Z_{l_i}^{(l_i)}\right)\right|^2$$
$$\times\left(\frac{\delta^k}{\prod_i \delta\left(\left|\Psi\left(\theta - \frac{\left|Z-Z_{l_i}^{(1)}\right|+...+\left|Z_{l_i}^{(l-1)}-Z_{l_i}^{(l_i)}\right|}{c}, Z_{l_i}^{(l_i)}\right)\right|^2\right)^{k_{l_i}}}\right)$$

Equation (176) can be shown recursively. To compute the $2n$ correlation functions, the subgraphs with $2k$ legs, $k<n$, are given by (176) at order $2k$. For $k=n$, the classical action yields a vertex:

$$\frac{1}{2}\left(\frac{\delta^n}{[n]!\delta^n\left|\Psi\right|^2}\left[\Psi^{\dagger}\left(\theta',Z\right)\nabla_{\theta}\left(\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^2\right)\Psi\left(\theta,Z\right)\right)\right]\right)_{\left|\Psi(\theta,Z)\right|^2=0}\left|\Psi\right|^{2n}$$

For $k>n$, a similar argument as in paragraph 1.1 in the vertex:

$$\frac{1}{2}\left(\frac{\delta^k}{[k]!\delta^k\left|\Psi\right|^2}\left[\Psi^{\dagger}\left(\theta',Z\right)\nabla_{\theta}\left(\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^2\right)\Psi\left(\theta,Z\right)\right)\right]\right)_{\left|\Psi(\theta,Z)\right|^2=0}\left|\Psi\right|^{2k}$$

$k-n$ factor $\left|\Psi\left(\theta,Z\right)\right|^2$ have to be replaced by $\mathcal{G}_0\left(0,0,Z\right)$. Summing over k, it means that the $2n$ vertex is computed with:

$$\frac{1}{2}\sum_{l=0}^{\infty}\left(\frac{\delta^{l+n}}{[l+n]!\delta^{l+n}\left|\Psi\right|^2}\left[\Psi^{\dagger}\left(\theta',Z\right)\nabla_{\theta}\left(\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^2\right)\Psi\left(\theta,Z\right)\right)\right]\right)_{\left|\Psi(\theta,Z)\right|^2=0}\left[C_{l+n}^{l}\right]\left(\mathcal{G}_0\left(0,0,Z\right)\right)^l\left|\Psi\right|^{2n}$$

where the symbol $\left[C_{l+n}^{l}\right]$ reminds that among the product $\left|\Psi\left(\theta_1,Z_1\right)\right|^2...\left|\Psi\left(\theta_{l+n},Z_{l+n}\right)\right|^2$ we sum over all the C_{l+n}^{l} possibilities to replace l factor $\left|\Psi\left(\theta_j,Z_j\right)\right|^2$ by $\mathcal{G}_0\left(0,0,Z_j\right)$. Summing the series, we find for the $2n$ vertices:

$$\begin{aligned}
&\frac{1}{2}\sum_{l=0}^{\infty}\left(\frac{\delta^{l+n}}{[l+n]!\delta^{l+n}\left|\Psi\right|^2}\left[\Psi^{\dagger}\left(\theta',Z\right)\nabla_{\theta}\left(\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^2\right)\Psi\left(\theta,Z\right)\right)\right]\right)_{\left|\Psi(\theta,Z)\right|^2=0}\left[C_{l+n}^{l}\right]\left(\mathcal{G}_0\left(0,0,Z\right)\right)^l\left|\Psi\right|^{2n}\\
=\ &\frac{1}{2}\left(\frac{\delta^n}{[n]!\delta^n\left|\Psi\right|^2}\left[\Psi^{\dagger}\left(\theta',Z\right)\nabla_{\theta}\left(\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^2\right)\Psi\left(\theta,Z\right)\right)\right]\right)_{\left|\Psi(\theta,Z)\right|^2=\mathcal{G}_0(0,Z)}\left|\Psi\right|^{2n}
\end{aligned}$$

as requested.

Below, to compute the higher order corrections to the effective potential, it will be useful to write (176) with an other set of variables. We replace:

$$\Psi\left(\theta-\frac{\left|Z-Z_{l_i}^{(1)}\right|+...+\left|Z_{l_i}^{(l-1)}-Z_{l_i}^{(l_i)}\right|}{c},Z_{l_i}^{(l_i)}\right)$$

by $\Psi\left(\theta-l_i,Z_i\right)$ where l_i represents an arbitrary delay time. As a consequence, the $2n$-th vertex:

$$V_{2n}=\left|\Psi\right|^{2n}\left(\frac{\delta^n}{[n]!\delta^n\left|\Psi\right|^2}\left[\Psi^{\dagger}\left(\theta',Z\right)\nabla_{\theta}\left(\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^2\right)\Psi\left(\theta,Z\right)\right)\right]\right)_{\left|\Psi(\theta,Z)\right|^2=\mathcal{G}_0(0,Z)}$$

becomes (where $\omega^{-1}(J,\theta,Z)$ stands for $\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)$ when no confusion is possible):

$$\begin{aligned}
V_{2n} &= \frac{1}{2(n)!}\left[\int \Psi^{\dagger}(\theta,Z)\frac{\delta^{n}\left[\int \Psi^{\dagger}(\theta,Z)\nabla_{\theta}\omega^{-1}(J,\theta,Z)\Psi(\theta,Z)\,dZd\theta\right]}{\prod_{i=1}^{n}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\right. \\
&\quad\left.\times\prod_{i=1}^{n-1}\left|\Psi(\theta-l_i,Z_i)\right|^2 dl_i\Psi(\theta,Z)\,dZd\theta\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)} \\
&= \frac{1}{2(n-1)!}\int \Psi^{\dagger}(\theta,Z)\nabla_{\theta}\left[\frac{\delta^{n-1}\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n-1}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\prod_{i=1}^{n-1}\left|\Psi(\theta-l_i,Z_i)\right|^2 dZ_i\Psi(\theta,Z)\,dZd\theta dl_i \\
&\quad+\frac{1}{2n!}\int\nabla_{\theta}\left[\frac{\delta^{n}\left(\nabla_{\theta}\omega^{-1}(J,\theta,Z)\mathcal{G}_0(\theta,\theta',Z)\right)_{\theta=\theta'}}{\prod_{i=1}^{n}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\prod_{i=1}^{n}\left|\Psi(\theta-l_i,Z_i)\right|^2\prod_{i=1}^{n}dZ_i dZd\theta dl_i
\end{aligned}\tag{177}$$

1.2.2 Estimation of (177)

Expression (177) can also be rewritten:

$$\begin{aligned}
V_{2n} &= \frac{1}{2}\int \Psi^{\dagger}(\theta,Z)\nabla_{\theta}\frac{\delta^{n-1}\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n-1}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\prod_{i=1}^{n-1}\left|\Psi(\theta-l_i,Z_i)\right|^2\prod_{i=1}^{n-1}dZ_i\Psi(\theta,Z)\,dZdl_i \\
&\quad+\int \mathcal{G}_0'(Z)\frac{\delta^{n}\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\prod_{i=1}^{n}\left|\Psi(\theta-l_i,Z_i)\right|^2\prod_{i=1}^{n}dZ_i dZdl_i \\
&\quad+\int \mathcal{G}_0(Z)\frac{\delta^{n}\nabla_{\theta}\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\prod_{i=1}^{n}\left|\Psi(\theta-l_i,Z_i)\right|^2\prod_{i=1}^{n}dZ_i dZdl_i
\end{aligned}\tag{178}$$

with:

$$\mathcal{G}_0'(Z)=\left(\frac{\nabla_{\theta}\mathcal{G}_0(\theta,\theta',Z)}{2}\right)_{\theta=\theta'}$$

However, the two last terms in (178) come from the backreaction of the n vertices on the whole system, and can be neglected in first approximation. Actually in a neighborhood of the permanent regime, we have:

$$\mathcal{G}_0(Z)\frac{\delta^{n}\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}<<\frac{\delta^{n-1}\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n-1}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}$$

The neglected terms will be reintroduced later. We can thus consider that:

$$V_{2n}=\frac{1}{2(n-1)!}\int \Psi^{\dagger}(\theta,Z)\nabla_{\theta}\frac{\delta^{n-1}\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n-1}\delta\left|\Psi(\theta-l_i,Z_i)\right|^2}\prod_{i=1}^{n-1}\left|\Psi(\theta-l_i,Z_i)\right|^2\prod_{i=1}^{n-1}dZ_i dl_i\Psi(\theta,Z)\,d\theta dZ\tag{179}$$

The neglected contributions will be reintroduced in Appendix 4.

The terms in (179) are the coefficients obtained by the expansion of $\omega^{-1}(J,\theta,Z)$ in powers of $\Psi^{\dagger}(\theta,Z)\Psi(\theta,Z)$. It is valid for $|\Psi(\theta,Z)|<1$. For $|\Psi(\theta,Z)|>1$, we can expand $\omega^{-1}(J,\theta,Z)$ in powers of $\frac{1}{|\Psi(\theta,Z)|}$. Given the form of F and since $\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)$, the expansion is obtained by replacing the derivatives of F by those of $-x^2F$ and by replacing ω with ω^{-1}.

Formula (179) yields the vertices V_{2n}, $n \leqslant N$, intervening in the computation of the $2N$ correlation functions. We have to estimate the derivatives arising in (179), before computing the effective action.

Appendix 2 General form of the graphs and convergence of the graphs expansions

We compute the sum of graphs involved in the partition function with source term, i.e. the graphs deduced from the.interaction terms (177). This is done in several step. We first give the general form of these graphs. Then, we compute the factors arising from the vertices (177). This allows to find the full sum of graphs, and to show its convergence.

2.1 General form of the graphs

The sum of graphs with $2n$-th external points is obtained by considering any graph between these points and made of 2 points propagators connected by vertices V_{2l} defined in (177), where $l \leqslant n$. The $2l$ vertices are given by:

$$V_{2l}\left(\left\{\left(\theta^{(k_i)}, Z_{k_i}\right)\right\}_{i=1,\ldots,l}\right) = \frac{1}{l!}\left[\frac{\delta^n\left[\int \Psi^{\dagger}(\theta, Z)\nabla_\theta\omega^{-1}(J,\theta,Z)\Psi(\theta,Z)\,dZd\theta\right]}{\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(k_i)}, Z_{k_i}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)} \tag{180}$$

To these vertices will be added the contributions of a stabilization potential. If we write this potential $V(\Psi)$, the vertex is modified as:

$$\begin{aligned}&\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)}, Z_{k_i}\right)\right\}_{i=1,\ldots,l}\right) \\ = &\frac{1}{l!}\left[\frac{\delta^l\left[\int \Psi^{\dagger}(\theta, Z)\nabla_\theta\omega^{-1}(J,\theta,Z)\Psi(\theta,Z)\,dZd\theta + \Psi^{\dagger}(\theta,Z)V(\Psi)\Psi(\theta,Z)\right]}{\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(k_i)}, Z_{k_i}\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\end{aligned} \tag{181}$$

The graphs have no loop drawn between two legs of any of the external points (these contributions are already taken into account by the expansion around $\mathcal{G}_0(0,Z) = \frac{1}{\Lambda}$). The absence of internal loops implies that the $2n$-th points graphs are made of n lines P_i. Each line P_i is associated to a point Z_i. It is drawn between an initial time $\theta_i^{(i)}$ and a final time $\theta_f^{(i)}$. We can thus write the $2n$-th external points as $\left(\theta_i^{(i)}, \theta_f^{(i)}, Z_i\right)_{i=1,\ldots,n}$ with $\theta_i^{(i)} < \theta_f^{(i)}$. The vertex $\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)}, Z_{k_i}\right)\right\}_{i=1,\ldots,l}\right)$ can be represented by a point (Z,θ) from which is issued $2l$ legs ending at the points $\left(\theta^{(k_i)}, Z_{k_i}\right)$. A graph thus consists in lines P_i that are cut by an arbitrary number of vertices of valence $l \leqslant n$ of the form $\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)}, Z_{k_i}\right)\right\}_{i=1,\ldots,l}\right)$ with $\theta_i^{(k_i)} < \theta^{(k_i)} < \theta_f^{(k_i)}$. We associate a propagator $\mathcal{G}_0(\theta,\theta',Z) = \frac{\exp\left(-\Lambda_1\left(\theta-\theta'\right)\right)}{\Lambda}H(\theta-\theta')$ to each segment of the graph between two vertices connected to the line labelled by Z at θ and θ', As in (177), the vertices $\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)},Z_{k_i}\right)\right\}_{i=1,...,l}\right)$ can be decomposed in two terms:

$$\frac{1}{2(n-1)!}\int\left[\frac{\delta^{l-1}\left(\nabla_\theta\omega^{-1}(J,\theta,Z)+V(\Psi)\right)}{\prod\limits_{i=1}^{l-1}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}\delta\left(\theta^{(k_i)}-\theta\right)\delta\left(Z_{k_i}-Z\right)dZd\theta dl_i$$
$$+\frac{1}{2n!}\int\left[\frac{\delta^{l}\left(\nabla_\theta\omega^{-1}(J,\theta,Z)\mathcal{G}_0(\theta,\theta',Z)+\mathcal{G}_0(\theta,\theta,Z)V(\Psi)\right)_{\theta=\theta'}}{\prod\limits_{i=1}^{n}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}dZd\theta$$

The second term computes the impact of the propagator $\mathcal{G}_0(\theta,\theta',Z)$ on the background Ψ, that is, on the whole system. As a consequence, its contribution can be neglected. Only the first term remains in first approximation and it is equivalent to constrain one of the derivatives in (181) to act on $\Psi^\dagger(\theta,Z)$ and $\Psi(\theta,Z)$ in the integral. We will compute the graphs in this approximation and account ultimately for the corrections due to the neglected terms.

We can picture picture the vertices (181) as a box cutting the lines P_{k_i}. The contributions associated to the segments between two vertices are 2 points Green functions $\mathcal{G}_0\left(\theta^{(k_i)},\theta^{(l_i)},Z_{k_i}\right)$. We can transform all the vertices of valence $2l\leqslant 2n$ as $2n$ points vertices. To do so, we define for $\{k_1,...,k_l\}\equiv\{k_i\}_l\subset\{1,...,n\}$ the $2n$ vertex:

$$\left[\hat{V}_{2n}^{\{k_i\}_l}\left(\left\{\left(\theta^{(m)},Z_m\right)\right\}_{m=1,...,n}\right)\right]=\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)},Z_{k_i}\right)\right\}_{i=1,...,l}\right)\otimes\left(\mathcal{G}_0^{-1}\right)^{\otimes\overline{\{k_i\}_{i=1,...,l}}}$$

where $\overline{\{k_i\}_{i=1,...,l}}$ is the complement of $\{k_i\}_{i=1,...,l}$ in $\{1,...,n\}$. The operators $\mathcal{G}_0^{-1}$ are local and depend on two variables:

$$\mathcal{G}_0^{-1}\equiv\mathcal{G}_0^{-1}\left(\theta^{(l_i)},Z_{l_i}\right)$$

Then:

$$\left(\mathcal{G}_0^{-1}\right)^{\otimes\overline{\{k_i\}_{i=1,...,l}}}=\prod_{l_i\subset\overline{\{k_i\}_{i=1,...,l}}}\mathcal{G}_0^{-1}\left(\theta^{(l_i)},Z_{l_i}\right)$$

The vertex $\left[\hat{V}_{2n}^{\{k_i\}_l}\left(\left\{\left(\theta^{(m)},Z_m\right)\right\}_{m=1,...,n}\right)\right]$ is represented by a box cutting all the lines P_i. For the variables $\left\{\left(\theta^{(k_i)},Z_{k_i}\right)\right\}_{i=1,...,l}$ the propagators on each side of the box are convoluted with $\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)},Z_{k_i}\right)\right\}_{i=1,...,l}\right)$. For the other variables, pairs of propagators are convoluted with their inverse, producing a single propagator, as needed. In the sequel, we write:

$$\left[\hat{V}_{2n}^{\{k_i\}_l}\left(\left\{\left(\theta^{(m)},Z_m\right)\right\}\right)\right]\equiv\left[\hat{V}_{2n}^{\{k_i\}_l}\left(\left\{\left(\theta^{(m)},Z_m\right)\right\}_{m=1,...,n}\right)\right]$$

and m runs implicitly from 1 to n.

As a consequence, the contribution of a graph made of an arbitrary sequence of vertices is:

$$\left[\mathcal{G}_0^{\otimes n}\left(\left(\theta^{(m_0)},\theta^{(m_1)},Z_{m_0}\right)\right)\right]*\left[\hat{V}_{2n}^{\{k_i^1\}_{l_1}}\left(\left\{\left(\theta^{(m_1)},Z_{m_1}\right)\right\}\right)\right]*\left[\mathcal{G}_0^{\otimes n}\left(\left(\theta^{(m_1)},\theta^{(m_2)},Z_{m_1}\right)\right)\right]$$
$$*...*\left[\hat{V}_{2n}^{\{k_i^p\}_{l_p}}\left(\left\{\left(\theta^{(m_{k-1})},Z_{m_{k-1}}\right)\right\}\right)\right]*\left[\mathcal{G}_0^{\otimes n}\left(\left(\theta^{(m_{p-1})},\theta^{(m_p)},Z_{m_p}\right)\right)\right]\tag{182}$$

with the constraint that $\theta^{(m_0)}=\theta_f^{(m_0)}$, $\theta^{(m_p)}=\theta_i^{(m_p)}$, and $\{Z_{m_i}\}=\{Z_m\}$ are fixed. The $2n$ points propagators are defined by the product of individual propagators:

$$\mathcal{G}_0^{\otimes n}\left(\left(\theta^{(m)},\theta^{(m')},Z_m\right)\right)\equiv\prod_{m=1}^{n}\mathcal{G}_0\left(\theta^{(m)},\theta^{(m')},Z_m\right)$$

The sum over all possible vertices is then:

$$\sum_p \frac{1}{p!}\left[\mathcal{G}_0^{\otimes n}\left(\left(\theta^{(m_0)},\theta^{(m_1)},Z_{m_0}\right)\right)\right] \tag{183}$$
$$\times\left(\sum_l\int\sum_{\{k_1,\ldots,k_l\}\subset\{1,\ldots,n\}}\left[\hat{V}_{2n}^{\{k_i\}_l}\left(\left\{\left(\theta^{(m_l)},Z_{m_l}\right)\right\}\right)\right]*\left[\mathcal{G}_0^{\otimes n}\left(\left(\theta^{(m_l)},\theta^{(m_l')},Z_{k_i}\right)\right)\right]\prod d\theta^{(m_l)}\right)^p$$

The $p!$ arises to avoid counting of equivalent graphs. The power is understood as successive convolutions.

As a consequence, the sum of graphs rewrites:

$$\mathcal{G}_0^{\otimes n}*\exp\left(\sum_{l=1}^n\sum_{\{k_1,\ldots,k_l\}\subset\{1,\ldots,n\}}\left[\hat{V}_{2n}^{\{k_i\}_l}*\mathcal{G}_0^{\otimes n}\right]\right)=\mathcal{G}_0^{\otimes n}*\exp\left(\left[\hat{V}_{2n}*\mathcal{G}_0^{\otimes n}\right]\right) \tag{184}$$

where $\left[\hat{V}_{2n}^{\{k_i\}_l}*\mathcal{G}_0^{\otimes n}\right]$ is the operator with kernel:

$$\left[\hat{V}_{2n}^{\{k_i\}_l}\left(\left\{\left(\theta^{(m_l)},Z_{m_l}\right)\right\}\right)\right]*\left[\mathcal{G}_0\left(\left(\theta^{(m_l)},\theta^{(m_l')},Z_{k_i}\right)\right)\right]$$

and where we defined:

$$\left[\hat{V}_{2n}\right]=\sum_{l=1}^n\sum_{\{k_1,\ldots,k_l\}\subset\{1,\ldots,n\}}\left[\hat{V}_{2n}^{\{k_i\}_l}\right]$$

Alternatively it is also given by:

$$\exp\left(\sum_{l=1}^n\sum_{\{k_1,\ldots,k_l\}\subset\{1,\ldots,n\}}\left[\mathcal{G}_0^{\otimes n}*\hat{V}_{2n}^{\{k_i\}_l}\right]\right)*\mathcal{G}_0^{\otimes n}=\exp\left(\left[\hat{V}_{2n}*\mathcal{G}_0^{\otimes n}\right]\right)*\mathcal{G}_0^{\otimes n} \tag{185}$$

We expand this formula in the next paragraph to show the convergence of the graph expansion. In turn, this proves the convergence of the one-particule irreducible graphs (1PI graphs) series expansion. The series expansion of 1PI graphs compute the effective action. Its precise form will be obtained by an other method in appendix 3.

2.2 Convergence of the series expansion

2.2.1 Expression of vertices arising in (184)

The series expansion of (184) has to be computed using the Wick theorem on the terms (182). Such terms are computed by inserting vertices (189) between propagators. These ones have the form between n pairs $\left(\theta_1^{(i)},\theta_2^{(i)},Z_i\right)_{i=1,\ldots,n}$:

$$\mathcal{G}_0\left(\left(\theta_1^{(i)},\theta_2^{(i)},Z_i\right)\right)=\frac{\exp\left(-\Lambda_1\left(\sum_{i=1}^n\theta_1^{(i)}-\sum_{i=1}^n\theta_2^{(i)}\right)\right)}{\Lambda} \tag{186}$$

with:

$$\Lambda = \sqrt{\frac{\pi}{2}}\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}$$
$$\Lambda_1 = \sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{1}{\sigma^2\bar{X}_r}$$

As a consequence, the convolution of m propagators leads to a global factor:

$$\frac{\exp\left(-\Lambda_1\left(\sum_{i=1}^n\theta_f^{(i)}-\sum_{i=1}^n\theta_i^{(i)}\right)\right)}{\Lambda^m}$$

The power m is given by the total number of vertices in the graph, each of them weighted by its valence. The vertices induce multiplicative factor at some times $\theta^{(i)}$ and integrations are performed over $\theta^{(i)}$. The presence of derivatives in the vertices induce terms of the form:

$$\frac{1}{2}\overline{\Psi^{\dagger}(\theta,Z)\nabla_{\theta}V(\theta,Z)\Psi(\theta,Z)}$$

where $V(\theta,Z)$ is any type of vertex and the upper bar denotes the contraction through Wick's theorem. It is equivalent to replace this term by:

$$\begin{aligned}\frac{1}{2}\nabla_{\theta}\left(V(\theta,Z)\mathcal{G}_0((\theta,\theta,Z))\right) &= \mathcal{G}_0((\theta,\theta,Z))\nabla_{\theta}V(\theta,Z)-\Lambda_1\mathcal{G}_0((\theta,\theta,Z))V(\theta,Z)\\ &= \frac{1}{\Lambda}\left(\nabla_{\theta}V(\theta,Z)-\Lambda_1 V(\theta,Z)\right)\end{aligned}$$

given (186). This means that in a sequence of propagators defining a graph, the vertex $\nabla_{\theta}V(\theta,Z)$ can be replaced by:

$$\begin{aligned}&\mathcal{G}_0^{-1}((\theta,\theta,Z))\left(\nabla_{\theta}\left(V(\theta,Z)\left(\mathcal{G}_0(\theta_1,\theta,Z)\right)\right)\right)_{\theta_1=\theta}\\ \equiv\ &\mathcal{G}_0^{-1}((0,Z))\left(\nabla_{\theta}\left(V(\theta,Z)\mathcal{G}_0\left(0^{+},Z\right)\right)\right)\end{aligned}$$

It implies that when a vertex is inserted at the left of a propagator, it can be replaced by:

$$\begin{aligned}&\hat{V}_{2l}\left(\left\{\left(\theta^{(k_i)},Z_{k_i}\right)\right\}_{i=1,\dots,l}\right) \qquad (187)\\ =&\left[\frac{\delta^l\left[\int\mathcal{G}_0^{-1}((0,Z))\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\omega^{-1}(J,\theta,Z)\mathcal{G}_0(0^{+},Z)\Psi(\theta,Z)\,dZd\theta+V(\Psi)\right]}{l!\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\\ \equiv&\left[\frac{\delta^l\left[\int\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\omega^{-1}(J,\theta,Z)\Psi(\theta,Z)\,dZd\theta+V(\Psi)\right]_{\mathcal{G}_0}}{l!\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}\end{aligned}$$

To compute the graphs series, the vertices (187) have to be expanded by taking into account the form of the potential V.

2.2.2 Expanded form of the vertices

As presented in the text, the potential for maintaining and activating new connections is chosen to be equal to:

$$\begin{aligned}V(\Psi) &= -\frac{\zeta_1}{2}\int\left(|\Psi(\theta,Z)|^2\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\right)\\ &\quad+\sum_{n=1}^{\infty}\frac{\zeta_n}{2n!}\int|\Psi(\theta,Z)|^2\left(\prod_{i=1}^{n-1}\left|\Psi\left(\theta-\frac{|Z-Z_i|}{c},Z_i\right)\right|^2\right) \qquad (188)\\ &= \sum_{n=2}^{\infty}\frac{1}{2n!}\zeta^{(n)}\int|\Psi(\theta,Z)|^2\left(\prod_{i=1}^{n-1}\left|\Psi\left(\theta-\frac{|Z-Z_i|}{c},Z_i\right)\right|^2\right)\end{aligned}$$

with:

$$\begin{aligned}\zeta^{(l)} &= \zeta_l,\ l>2\\ \zeta^{(2)} &= \zeta_2-\zeta_1\\ \zeta^{(1)} &= 0\end{aligned}$$

The second term represents the limitation in increasing the number of connections. This amounts to shift the vertices by $+\zeta_2$. The factor $-\zeta_1$ accounts for a minimal number of connections maintained. It depends on external activity J.

The first term modifies the 4-th vertices by $-\zeta_1$. The vertices involved in the $2n$ points correlation function are given by an expansion of $V(\Psi)$ around $\mathcal{G}_0(0,Z)$. The first order expansion in $|\Psi(\theta,Z)|^2$ modifies the 2 points propagator by replacing α with:

$$\alpha+\sum_{k\geqslant 2}\frac{1}{k!}C_k^1\frac{\zeta^{(k)}}{\Lambda^{k-1}}=\sum_{k\geqslant l}\frac{1}{(k-1)!}\frac{\zeta^{(k)}}{\Lambda^{k-1}}$$

which modifies the values of Λ and Λ_1.

For $n\geqslant 2$, the $2n$ vertex is then:

$$\begin{aligned}\hat{V}_{2l}\left(\left(\theta^{(i)},Z_i\right)_{i=1,\dots,n}\right) &= \frac{1}{2(l)!}\left[\frac{\delta^l\left[\int\Psi^\dagger(\theta,Z)\nabla_\theta\omega^{-1}(J,\theta,Z)\Psi(\theta,Z)\,dZd\theta\right]}{\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}} \\ &\quad -\left[\frac{\delta^l\left(\sum_{k=l}^{\infty}\frac{1}{2k!}\zeta^{(k)}\int|\Psi(\theta,Z)|^2\left(\prod_{i=1}^{k-1}\left|\Psi\left(\theta-\frac{|Z-Z_i|}{c},Z_i\right)\right|^2\right)\right)}{\prod_{j=1}^{l}\delta\left|\Psi(\theta_j,Z_j)\right|^2}\right]_{\substack{|\Psi(\theta_j,Z_j)|^2\\=\mathcal{G}_0(0,Z_j)}}\end{aligned}\tag{189}$$

To do so, we decompose (189) in two types of vertices for n given points $\left(\theta^{(1)},Z_1\right),\dots,\left(\theta^{(n)},Z_n\right)$:

$$\begin{aligned}&\hat{V}_{2l}^{(1)}\left(\left(\theta^{(i)},Z_i\right),\left\{Z_j,\theta^{(j)}\right\}_{j\neq i}\right)\\ =\;&\frac{1}{2(l)!}\sum_{i=1}^{l}\left[\frac{\left[\int\Psi^\dagger\left(\theta^{(i)},Z_i\right)\delta^{l-1}\left[\nabla_\theta\omega^{-1}\left(J,\theta^{(i)},Z_i\right)\right]\Psi\left(\theta^{(i)},Z_i\right)\right]_{\mathcal{G}_0}}{\prod_{j=1,j\neq i}^{l}\delta\left|\Psi\left(\theta^{(j)},Z_j\right)\right|^2}\right]_{|\Psi(\theta_j,Z_j)|^2=\mathcal{G}_0(0,Z_j)}\\ &-\frac{1}{2(l)!}\sum_{i=1}^{l}\left[\frac{\delta^{l-1}\left[\sum_{k=l}^{\infty}\frac{1}{k!}\zeta^{(k)}\left(\prod_{i=1}^{k-1}\left|\Psi\left(\theta-\frac{|Z-Z_i|}{c},Z_i\right)\right|^2\right)\right]}{\prod_{j=1,j\neq i}^{l}\delta\left|\Psi\left(\theta^{(j)},Z_j\right)\right|^2}\right]_{|\Psi(\theta_j,Z_j)|^2=\mathcal{G}_0(0,Z_j)}\end{aligned}$$

and:

$$\begin{aligned}\hat{V}_{2l}^{(2)}\left(\left(\theta^{(i)},Z_i\right)_{i=1,\dots,n}\right) &= \frac{1}{2(l)!}\left[\frac{\left[\int\Psi^\dagger(\theta,Z)\nabla_\theta\delta^l\left[\omega^{-1}(J,\theta,Z)\right]\Psi(\theta,Z)\,dZd\theta\right]_{\mathcal{G}_0}}{\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)} \\ &\quad -\left[\int\mathcal{G}_0(0,Z)\,dZd\theta\frac{\delta^l\left[\sum_{k=l}^{\infty}\frac{1}{2k!}\zeta^{(k)}\left(\prod_{i=1}^{k-1}\left|\Psi\left(\theta-\frac{|Z-Z_i|}{c},Z_i\right)\right|^2\right)\right]}{\prod_{j=1}^{l}\delta\left|\Psi(\theta_j,Z_j)\right|^2}\right]_{\substack{|\Psi(\theta_j,Z_j)|^2\\=\mathcal{G}_0(0,Z_j)}}\end{aligned}\tag{190}$$

As explained before, the index $\mathcal{G}_0$ denotes the contraction between the field on the left with the one on the right when the vertex has a propagator on its right. It is equivalent to introduce $\mathcal{G}_0(0,Z)$ inside the gradient, to remove $\Psi^\dagger(\theta,Z)$ and $\Psi(\theta,Z)$ and multiply by $\mathcal{G}_0^{-1}(0,Z)$.

To complete the computation of the vertices $\hat{V}_{2l}^{(1)}$ and $\hat{V}_{2l}^{(2)}$ we can regroup the terms involving the coefficients of the potential V. To do so, we simplify the derivatives of the potential by writing:

$$\sum_{i=1}^{l}\left[\sum_{k=l}^{\infty}\frac{1}{2k!}\zeta^{(k)}\frac{\delta^{l-1}}{\prod\limits_{j=1,j\neq i}^{l}\delta\left|\Psi\left(\theta_{j},Z_{j}\right)\right|^{2}}\left(\prod_{i=1}^{k-1}\left|\Psi\left(\theta-\frac{|Z-Z_{i}|}{c},Z_{i}\right)\right|^{2}\right)\right]_{\substack{|\Psi(\theta_{j},Z_{j})|^{2}\\=\mathcal{G}_{0}(0,Z_{j})}}$$

$$+\left[\sum_{k=l}^{\infty}\frac{1}{2k!}\zeta^{(k)}\int\mathcal{G}_{0}\left(0,Z\right)dZd\theta\frac{\delta^{l}\left[\left(\prod\limits_{i=1}^{k-1}\left|\Psi\left(\theta-\frac{|Z-Z_{i}|}{c},Z_{i}\right)\right|^{2}\right)\right]}{\prod\limits_{j=1}^{l}\delta\left|\Psi\left(\theta_{j},Z_{j}\right)\right|^{2}}\right]_{\substack{|\Psi(\theta_{j},Z_{j})|^{2}\\=\mathcal{G}_{0}(0,Z_{j})}}$$

$$=\sum_{i=1}^{l}\sum_{k=l}^{\infty}\frac{1}{2k!}C_{k-1}^{l-1}\frac{\zeta^{(k)}}{\Lambda^{k-l}}\prod_{j=1,j\neq i}^{l}\delta\left(\theta_{i}-\frac{|Z-Z_{i}|}{c}-\theta_{j}\right)$$

$$+\sum_{k=l+1}^{\infty}\frac{1}{2k!}\zeta^{(k)}\int\mathcal{G}_{0}\left(\theta,\theta,Z\right)C_{k-1}^{l}\prod_{i=1}^{l}\delta\left(\theta-\frac{|Z-Z_{i}|}{c}-\theta_{i}\right)$$

$$=\frac{1}{2\left(l-1\right)!}\sum_{k=l}^{\infty}\frac{1}{k\left(k-l\right)!}\frac{\zeta^{(k)}}{\Lambda^{k-l}}\prod_{j=1,j\neq i}^{l}\delta\left(\theta_{i}-\frac{|Z-Z_{i}|}{c}-\theta_{j}\right)$$

$$+\frac{1}{2l!}\sum_{k=l}^{\infty}\int\frac{1}{k\left(k-l-1\right)!}\frac{\zeta^{(k)}}{\Lambda^{k-l-1}}\prod_{i=1}^{l}\delta\left(\theta-\frac{|Z-Z_{i}|}{c}-\theta_{i}\right)$$

$$=\frac{1}{2\left(l-1\right)!}\zeta_{e}^{(l)}\prod_{j=1,j\neq i}^{l}\delta\left(\theta_{i}-\frac{|Z-Z_{i}|}{c}-\theta_{j}\right)+\frac{1}{2l!}\int\zeta_{e}^{(l+1)}\prod_{i=1}^{l}\delta\left(\theta-\frac{|Z-Z_{i}|}{c}-\theta_{i}\right)$$

with:

$$\zeta_{e}^{(l)}=\sum_{k\geqslant l}\frac{1}{k\left(k-l\right)!}\frac{\zeta^{(k)}}{\Lambda^{k-l}}\tag{191}$$

For $\zeta^{(k)}$ slowly varying and $\Lambda>>1$, this is approximatively equal to $\zeta^{(l)}$. We keep the notation $\zeta_{e}^{(l)}\rightarrow\zeta^{(l)}$.

As a consequence, $\hat{V}_{2l}^{(1)}$ and $\hat{V}_{2l}^{(2)}$ write:

$$\sum_{i=1}^{n}\hat{V}_{2l}^{(1)}\left(\left(\theta^{(i)},Z_{i}\right),\left\{Z_{j},\theta^{(j)}\right\}_{j\neq i}\right)\tag{192}$$

$$=\sum_{i=1}^{n}\left[\frac{1}{2\left(l\right)!}\frac{\delta^{l-1}\left[\nabla_{\theta}\omega^{-1}\left(J,\theta^{(i)},Z_{i}\right)\mathcal{G}_{0}\left(0,Z_{i}\right)\right]}{\mathcal{G}_{0}\left(0,Z_{i}\right)\prod\limits_{j=1,j\neq i}^{l}\delta\left|\Psi\left(\theta^{(j)},Z_{j}\right)\right|^{2}}-\frac{1}{2\left(l-1\right)!}\zeta^{(l)}\prod_{j=1,j\neq i}^{l}\delta\left(\theta_{i}-\frac{|Z-Z_{i}|}{c}-\theta_{j}\right)\right]_{\substack{|\Psi(\theta_{j},Z_{j})|^{2}\\=\mathcal{G}_{0}(0,Z_{j})}}$$

and:

$$\hat{V}_{2l}^{(2)}\left(\left(\theta^{(i)},Z_{i}\right)_{i=1,\ldots,n}\right)\tag{193}$$

$$=\frac{1}{2\left(l\right)!}\left[\int\nabla_{\theta}\frac{\delta^{l}\left[\omega^{-1}\left(J,\theta,Z\right)\right]}{\prod\limits_{i=1}^{l}\delta\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}}\mathcal{G}_{0}\left(0,Z\right)dZd\theta-\int\zeta^{(l+1)}\prod_{i=1}^{l}\delta\left(\theta-\frac{|Z-Z_{i}|}{c}-\theta_{i}\right)\right]_{\substack{|\Psi(\theta,Z)|^{2}\\=\mathcal{G}_{0}(0,Z)}}$$

with $\zeta_e^{(l)}$ given by (191). For $\zeta^{(k)}$ slowly varying and $\Lambda >> 1$, this is approximatively equal to $\zeta^{(l)}$. For the sake of simplicity we keep the notation $\zeta_e^{(l)} \rightarrow \zeta^{(l)}$.

As explained before, the vertex $\hat{V}_{2l}^{(2)}\left(\left(\theta^{(i)}, Z_i\right)_{i=1,\ldots,n}\right)$ can be neglected with respect to $\hat{V}_{2l}^{(1)}\left(\left(\theta^{(i)}, Z_i\right)_{i=1,\ldots,n}\right)$ in first approximation. We will compute the graphs associated to $\hat{V}_{2l}^{(1)}\left(\left(\theta^{(i)}, Z_i\right)_{i=1,\ldots,n}\right)$ in the next paragraph, and will then compute the contributions due to $\hat{V}_{2l}^{(2)}\left(\left(\theta^{(i)}, Z_i\right)_{i=1,\ldots,n}\right)$ as corrections.

Ultimately, and for later purpose, we also define:

$$\bar{\zeta}_n = \sum_{l=2}^{n} \sum_{\{k_1,\ldots,k_{l-1}\}\subset\{1,\ldots,n-1\},k_j\neq i} \frac{\zeta^{(l)}}{\Lambda^l} = \sum_{l=1}^{n} C_{n-1}^{l-1} \frac{\zeta^{(l)}}{\Lambda^l}$$

For example:

$$\bar{\zeta}_2 = \frac{\zeta^{(2)}}{\Lambda}, \bar{\zeta}_3 = \frac{\zeta^{(3)}}{\Lambda^2} + 3\frac{\zeta^{(2)}}{\Lambda}$$

If we express $\bar{\zeta}_n$ as a function of the initial set of variables $\zeta^{(l)}$, we have:

$$\bar{\zeta}_n = \sum_{l=1}^{n} C_n^l \frac{\sum_{k\geqslant l} \frac{1}{(k-l)!} \frac{\zeta^{(k)}}{\Lambda^{k-l}}}{\Lambda^{l-1}} = \sum_{l=1}^{n} C_n^l \sum_{k\geqslant l} \frac{1}{(k-l)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}}$$

so that:

$$\bar{\zeta}_2 = \sum_{k\geqslant 2} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} = \frac{\zeta^{(2)}}{\Lambda} + \sum_{k\geqslant 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}}$$

and:

$$\begin{aligned}
\bar{\zeta}_3 &= \sum_{k\geqslant 3} \frac{1}{(k-3)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k\geqslant 2} \frac{3}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} \\
&= 3\frac{\zeta^{(2)}}{\Lambda} + 3\sum_{k\geqslant 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k\geqslant 3} \frac{1}{(k-3)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} \\
&= 3\frac{\zeta^{(2)}}{\Lambda} + 3\sum_{k\geqslant 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k\geqslant 3} \frac{Sup\left(1, (k-3)\right) \zeta^{(k)}}{(k-2)!\Lambda^{k-1}}
\end{aligned}$$

We will assume that $\bar{\zeta}_2 < 0$ and $\bar{\zeta}_n > 0$ for $n > 2$. This is possible under the conditions:

$$\sum_{k\geqslant 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} < \frac{\left|\zeta^{(2)}\right|}{\Lambda} < \sum_{k\geqslant 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} + \sum_{k\geqslant 3} \frac{Sup\left(1, (k-2)\right) \zeta^{(k)}}{3\,(k-2)!\Lambda^{k-1}}$$

that are satisfied for a certain range of the parameters, since:

$$\sum_{k\geqslant 3} \frac{Sup\left(1, (k-2)\right) \zeta^{(k)}}{3\,(k-2)!\Lambda^{k-1}} - \sum_{k\geqslant 3} \frac{1}{(k-2)!} \frac{\zeta^{(k)}}{\Lambda^{k-1}} = \sum_{k\geqslant 3} \frac{\left(Sup\left(1, (k-2)\right) - 3\right) \zeta^{(k)}}{3\,(k-2)!\Lambda^{k-1}}$$

is positive for $\zeta^{(k)}$ large enough for $k > 5$.

2.2.3 Integral form of (184)

The expansion of (184) can be performed using the previous results. Each vertex of valence v can be attributed a factor $\frac{1}{\Lambda^{v}}$. In (192) we also change the variables $\theta^{(j)}=\theta^{(i)}-l_i$. We also define:

$$\bar{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)=\frac{\sum_{\substack{\{k_1,...,k_{l-1}\}\\ \subset\{1,...,n-1\}}}\prod_{k_j}\int_{\frac{\left|Z_i-Z_{k_j}\right|}{c}}^{\theta^{(i)}-\theta_i^{(k_j)}}dl_{k_j}}{2\Lambda^l}\times\left[\frac{\delta^{l-1}\left[\nabla_\theta\omega^{-1}\left(J,\theta^{(i)},Z_i\right)\mathcal{G}_0\left(0,Z_i\right)\right]}{\mathcal{G}_0\left(0,Z_i\right)\prod_{j=1,j\neq i}^{l}\delta\left|\Psi\left(\theta^{(i)}-l_{k_j},Z_{k_j}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)} \tag{194}$$

$$\hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)=\bar{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)-\frac{\zeta^{(l)}}{\Lambda^l} \tag{195}$$

$$\bar{\Lambda}_1^{(l)}\left(Z,\theta,\left\{Z_i,\theta^{(i)}\right\}_{i=1,...l}\right)=\frac{1}{2\left(l\right)!\Lambda^l}\left[\int\nabla_\theta\frac{\delta^l\left[\omega^{-1}\left(J,\theta,Z\right)\right]}{\prod_{i=1}^{l}\delta\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2}\mathcal{G}_0\left(0,Z\right)dZd\theta\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)} \tag{196}$$

and:

$$\hat{\Lambda}_1^{(l)}\left(Z,\theta,\left\{Z_i,\theta^{(i)}\right\}_{i=1,...l}\right)=\bar{\Lambda}_1^{(l)}\left(Z,\theta,\left\{Z_i,\theta^{(i)}\right\}_{i=1,...l}\right)-\frac{\zeta^{(l+1)}}{\Lambda^l}$$

The propagators induced by the vertices in (??) have been included in the definition of $\bar{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)$. The functions depend implicitely on the border of the timespans $\left[\theta_f^{(j)},\theta_i^{(j)}\right]$. Actually, the integrations $\int_{\frac{\left|Z_i-Z_{k_j}\right|}{c}}^{\theta^{(i)}-\theta_i^{(k_j)}}dl_{k_j}$ induces the presence of products of Heaviside functions $H\left(\theta_f^{(i)}-\theta_i^{(k_j)}-\frac{\left|Z_i-Z_{k_j}\right|}{c}\right)$.

We first compute the full sum of graphs arising from all combination of vertices $\hat{V}_{2l}^{(1)}$ defined in (192) between n initial points and n final points. The vertices $\hat{V}_{2l}^{(2)}$ (see (193)) will be included later. The vertices $\hat{V}_{2l}^{(1)}\left(\left(\theta^{(i)},Z_i\right),\left\{Z_j,\theta^{(j)}\right\}_{j\neq i}\right)$ can be associated to an initial point $\left(\theta^{(i)},Z_i\right)$ with $l-1$ final points among the $n-1$ others.

The factors associated to the vertices in the expansion of (184) have been found in the previous paragraph. We aasociate a global factor $\frac{\exp\left(-\Lambda_1\left(\sum_{i=1}^n\theta_f^{(i)}-\sum_{i=1}^n\theta_i^{(i)}\right)\right)}{\Lambda^m}$ to a sequence of vertices. Then the insertion of vertices at $\left(\theta^{(i)},Z_i\right)$, the sum over the final points and the integration over the $\left\{Z_j,\theta^{(j)}\right\}_{j\neq i}$ leads to a factor:

$$\frac{1}{\Lambda^l}\sum_{\substack{\{k_1,...,k_{l-1}\}\\ \subset\{1,...,n-1\}}}\int\hat{V}_{2l}^{(1)}\left(\left(\theta^{(i)},Z_i\right),\left\{Z_j,\theta^{(k_j)}\right\}_{k_j\neq i}\right)d\theta^{(k_j)}=\hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)$$

For each line associated to Z_i, the insertion of $\sum_{l=1}^n k_l^{(i)}$ vertices where $k_l^{(i)}$ vertices have valence l implies the integration over $\theta_i^{(i)}<\theta_1^{(i)}<...\theta_{\sum_{l=1}^n k_l^{(i)}}^{(i)}<\theta_f^{(i)}$ of the product of terms $\hat{\Xi}_1^{(l_q)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)$ for $q=1$ to $\sum_{l=1}^n k_l^{(i)}$. The number of l_q equal to l is $k_l^{(i)}$. Once an order $l_1,l_2,...$ is chosen, there are $\prod k_l^{(i)}!$ ways to order the vertices satisfying this order. Then, summing over the various orders $l_1,l_2,...$ and over the $k_l^{(i)}$ such that $\sum_{l=1}^n k_l^{(i)}=m$ is fixed, the global factor associated to the vertices is:

$$\int_{\theta_i^{(i)}<\theta_1^{(i)}<...\theta_m^{(i)}<\theta_f^{(i)}}\prod_{q=1}^{m}\left(\sum_{l=2}^{n}\hat{\Xi}_1^{(l_q)}\left(Z_i,\theta_q^{(i)},\{Z_j\}_{j\neq i}\right)\right)\delta\left(\theta_1^{(i)}-\theta_i^{(i)}\right)\delta\left(\theta_m^{(i)}-\theta_f^{(i)}\right)d\theta_q^{(i)} \tag{197}$$

The delta functions accounts for the fact that without external legs, two vertices are set at the borders of the interval. If we approximate $\sum_l \hat{\Xi}_1^{(l)}\left(Z_i,\theta_q^{(i)},\{Z_j\}_{j\neq i}\right)$ by its average on the interval $\left[\theta_i^{(i)},\theta_f^{(i)}\right]$, that is:

$$\frac{\int_{\theta_i^{(i)}}^{\theta_f^{(i)}}\sum_{l=2}^n \hat{\Xi}_1^{(l)}\left(Z_i,\theta_q^{(i)},\{Z_j\}_{j\neq i}\right)d\theta_q^{(i)}}{\theta_f^{(i)}-\theta_i^{(i)}}$$

As a consequence (197) becomes:

$$\begin{aligned}&\left(\frac{\int_{\theta_i^{(i)}}^{\theta_f^{(i)}}\sum_{l=2}^n \hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)d\theta^{(i)}}{\theta_f^{(i)}-\theta_i^{(i)}}\right)^m\int_{\theta_i^{(i)}<\theta_1^{(i)}<\ldots\theta_m^{(i)}<\theta_f^{(i)}}\delta\left(\theta_1^{(i)}-\theta_i^{(i)}\right)\delta\left(\theta_m^{(i)}-\theta_f^{(i)}\right)\prod_{q=1}^m d\theta_q^{(i)}\\ =\ &\frac{1}{m!}\left(\int_{\theta_i^{(i)}}^{\theta_f^{(i)}}\sum_{l=2}^n \hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)d\theta^{(i)}\right)^m\end{aligned}$$

To obtain the expansion (184) due to the vertices at Z_i, we sum over m and we multiply with a free propagator on the left which amounts to introduce a factor $\frac{1}{\Lambda}$ which leads to a contribution:

$$\frac{1}{\Lambda}\exp\left(\int_{\theta_i^{(i)}}^{\theta_f^{(i)}}\sum_{l=2}^n \hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)d\theta^{(i)}\right)$$

The full sum of graphs is then obtained by taking the product over i of these contributions, introducing the global factor $\exp\left(-\Lambda_1\left(\sum_{i=1}^n\theta_f^{(i)}-\sum_{i=1}^n\theta_i^{(i)}\right)\right)$ and suming over n. The sum of graphs is thus:

$$\begin{aligned}&\frac{\exp\left(-\Lambda_1\left(\sum_{i=1}^n\theta_f^{(i)}-\sum_{i=1}^n\theta_i^{(i)}\right)\right)}{\Lambda^n}\exp\left(\sum_i\int_{\theta_i^{(i)}}^{\theta_f^{(i)}}\sum_l \hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)d\theta^{(i)}\right)\\ =\ &\frac{\exp\left(-\Lambda_1\left(\sum_{i=1}^n\theta_f^{(i)}-\sum_{i=1}^n\theta_i^{(i)}\right)\right)}{\Lambda^n}\exp\left(\sum_i\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)\right)\end{aligned}$$

with:

$$\begin{aligned}\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)&=\int_{\theta_i^{(i)}}^{\theta_f^{(i)}}\sum_{l=2}^n \hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)d\theta^{(i)}\\&=\sum_{l=2}^n\sum_{\{k_1,\ldots,k_l\}\subset\{1,\ldots,n\},k_j\neq i}\hat{\Xi}_1^{(l)}\left(Z_i,\{Z_{k_j}\},\theta_i^{(i)},\theta_f^{(i)}\right)\end{aligned}\tag{198}$$

or alternatively:

$$\begin{aligned}\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)&=\bar{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)-\bar{\zeta}_n\left(\theta_f^{(i)}-\theta_i^{(i)}\right)\\&=\sum_{l=2}^n\sum_{\{k_1,\ldots,k_{l-1}\}\subset\{1,\ldots,n-1\},k_j\neq i}\left(\bar{\Xi}_1^{(l)}\left(Z_i,\{Z_{k_j}\},\theta_i^{(i)},\theta_f^{(i)}\right)-\frac{\zeta^{(l)}\left(\theta_f^{(i)}-\theta_i^{(i)}\right)}{\Lambda^l}\right)\end{aligned}$$

2.2.4 Contributions of $\hat{V}_{2l}^{(2)}\left(\left(\theta^{(i)},Z_i\right)_{i=1,\ldots,n}\right)$

We can include the contributions due to the vertices $\hat{V}_{2l}^{(2)}\left(\left(\theta^{(i)},Z_i\right)_{i=1,\ldots,n}\right)$ defined in (181) to the sum of graphs. These vertices are inserted at some times $\theta_1<\ldots<\theta_n$ and at each insertion θ_k one has

$\theta_i^{(i)} < \theta_k^{(i)} < \theta_k$. As before this leads to an overall factor:

$$\exp\left(\sum_{\substack{\{k_1,\ldots,k_l\}\\ \subset\{1,\ldots,n\}}}\int\prod_{i=1}^{l}\int_{\theta_i^{(k_i)}}^{\theta} d\theta^{(k_i)}\hat{\Lambda}_1^{(l)}\left(Z,\theta,\left\{Z_{k_i},\theta^{(k_i)}\right\}_{i=1,\ldots l}\right)d\theta\right) = \exp\left(\hat{\Lambda}_{1,n}\left(\left\{Z_i,\theta_i^{(i)},\theta_f^{(i)}\right\}\right)\right) \tag{199}$$

As a consequence, the sum (184) of graphs with $2n$ external points becomes:

$$\begin{aligned}&\frac{\exp\left(-\Lambda_1\left(\sum_{i=1}^n\theta_f^{(i)}-\sum_{i=1}^n\theta_i^{(i)}\right)\right)}{\Lambda^n}\\ &\times\exp\left(\sum_i\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)\right)\exp\left(\hat{\Lambda}_{1,n}\left(\left\{Z_i,\theta_i^{(i)},\theta_f^{(i)}\right\}\right)\right)\end{aligned} \tag{200}$$

Given (194)

$$\hat{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right) = \bar{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right) - \frac{\zeta^{(l)}}{\Lambda^l}$$

and (198):

$$\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right) = \sum_{l=2}^{n}\sum_{\{k_1,\ldots,k_l\}\subset\{1,\ldots,n\},k_j\neq i}\hat{\Xi}_1^{(l)}\left(Z_i,\left\{Z_{k_j}\right\},\theta_i^{(i)},\theta_f^{(i)}\right)$$

we can estimate in average the magnitude of $\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)$:

$$\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right) \simeq \sum_{l=2}^{n}C_n^l\left(\frac{\zeta^{(l)}}{\Lambda^l}\right)$$

if $\bar{\Xi}_1^{(l)}\left(Z_i,\theta^{(i)},\{Z_j\}_{j\neq i}\right)$ and $\zeta^{(l)}$ are of the same order. For $\zeta^{(l)}$ decreasing faster than $\exp(-l)$, for example $\zeta^{(l)} \simeq \exp(-l^\alpha)$ with $\alpha > 1$, the sum is converging. Actually, writing $C_n^l \simeq \exp(n\ln n - (n-l)\ln(n-l) - l\ln l)$, we have

$$C_n^l\left(\frac{\zeta^{(l)}}{\Lambda^l}\right) \simeq \exp\left(-n\left(x\ln x + (1-x)\ln(1-x) + n^{\alpha-1}x^\alpha + x\ln\Lambda\right)\right)$$

with $x = \frac{l}{n}$. As a consequence:

$$\sum_{l=2}^{n}C_n^l\left(\frac{\zeta^{(l)}}{\Lambda^l}\right) \simeq \int_0^1\exp\left(-n\left(x\ln x + (1-x)\ln(1-x) + n^{\alpha-1}x^\alpha + x\ln\Lambda\right)\right)dx$$

and the integral converges for $n\to\infty$. For $\zeta^{(2)} << 1$ we thus have $\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right) << 1$ and for slowly varying parameters, we can replace $\hat{\Xi}_{1,n}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)$ by its limit $\hat{\Xi}_{1,\infty}\left(Z_i,\{Z_j\}_{j\neq i},\theta_i^{(i)},\theta_f^{(i)}\right)$.

2.2.5 Convergence of the graph expansion (184)

An estimation of the expansion of graphs of order higher than 2 is given by:

$$\sum\frac{1}{n!}\int\prod_{i=1}^{n}\Psi^\dagger\left(\theta_i^{(i)},Z_i\right)\exp\left(\left[\bar{V}_{2n}\right]\right)\left[\bar{V}_{2n}\right]\left[\bar{V}_{2n}\right]\prod_{i=1}^{n}\Psi\left(\theta_i^{(i)},Z_i\right)$$

where all the graphs are taken into account. Given our previous results, this is equal to:

$$\begin{aligned}
&\simeq \sum \frac{1}{n!} \int \prod_{i=1}^{n} \Psi^{\dagger}\left(\theta_i^{(i)}, Z_i\right) \exp\left(\hat{\Xi}_{1,\infty}(Z_i)\right) \left(\hat{\Xi}_{1,\infty}(Z_i)\right)^2 \prod_{i=1}^{n} \Psi\left(\theta_i^{(i)}, Z_i\right) \\
&\simeq \hat{\Xi}_{1,\infty}\hat{\Xi}_{1,\infty} \sum \frac{1}{n!} \int \prod_{i=1}^{n} \Psi^{\dagger}\left(\theta_i^{(i)}, Z_i\right) \exp\left(\hat{\Xi}_{1,\infty}(Z_i)\right) \prod_{i=1}^{n} \Psi\left(\theta_i^{(i)}, Z_i\right) \\
&= \left(\hat{\Xi}_{1,\infty}\right)^2 \exp\left(\int \Psi^{\dagger}\left(\theta_i^{(i)}, Z_i\right) \exp\left(\hat{\Xi}_{1,\infty}(Z_i)\right) \Psi\left(\theta_i^{(i)}, Z_i\right)\right)
\end{aligned}$$

where $\hat{\Xi}_{1,\infty}$ is the average of $\hat{\Xi}_{1,\infty}(Z_i)$ over the thread. The perturbative expansion in $\hat{\Xi}_{1,\infty}$ is thus convergent.

Appendix 3. Estimation of the effective action and its minimum

The previous section showed the convergence of the full graphs series expansion. To find the effective action we have to restrict the sum to the 1PI graphs. Once the effective action will be found, we will write the equation of its minimum and compute the background field.

3.1 Effective action at the lowest order

We have seen in equation (174) that the 2 points Green function are computed using the action:

$$\begin{aligned}
\Gamma_0\left(\Psi^{\dagger}, \Psi\right) &\equiv -\frac{1}{2} \int \Psi^{\dagger}(\theta, Z) \left(\nabla_\theta \frac{\sigma_\theta^2}{2} \nabla_\theta\right) \Psi(\theta, Z) \\
&\quad + \frac{1}{2} \int \Psi^{\dagger}(\theta, Z) \left[\frac{\delta\left[\Psi^{\dagger}(\theta, Z) \nabla_\theta \left(\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right) \Psi(\theta, Z)\right)\right]}{\delta |\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2 \\ =\mathcal{G}_0(0,Z)}} \Psi(\theta, Z) \\
&\quad + \frac{1}{2} \int \Psi^{\dagger}(\theta, Z) \left[\frac{\delta\left[V(\Psi)\right]}{\delta |\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2 \\ =\mathcal{G}_0(0,Z)}} \Psi(\theta, Z) \\
&= \Psi^{\dagger}(\theta, Z) \left[\frac{\delta\left[S_{cl}\left(\Psi^{\dagger}, \Psi\right)\right]}{\delta |\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2 \\ =\mathcal{G}_0(0,Z)}} \Psi(\theta, Z)
\end{aligned}$$

with:

$$\begin{aligned}
S_{cl}\left(\Psi^{\dagger}, \Psi\right) &= -\frac{1}{2} \Psi^{\dagger}(\theta, Z) \left(\nabla_\theta \left(\frac{\sigma_\theta^2}{2} \nabla_\theta - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\right) \Psi(\theta, Z) \\
&+\alpha \int \left|\Psi\left(\theta^{(i)}, Z_i\right)\right|^2 + V(\Psi)
\end{aligned} \tag{201}$$

and:

$$\left[\frac{\delta\left[\Psi^{\dagger}(\theta, Z) \nabla_\theta \left(\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right) \Psi(\theta, Z)\right)\right]}{\delta |\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2 \\ =\mathcal{G}_0(0,Z)}}$$

defined as:

$$\left[\frac{\delta\left[\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\left(\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\Psi(\theta,Z)\right)\right]}{\delta|\Psi|^{2}}\right]_{\substack{|\Psi(\theta,Z)|^{2}\\=\mathcal{G}_{0}(0,Z)}}$$

$$= \omega^{-1}\left(\mathcal{G}_{0}(0,Z)\right)\nabla_{\theta}+\left(\nabla_{\theta}\left(\frac{\delta\left[\omega^{-1}\left(\bar{J},Z,\mathcal{G}_{0}\right)\right]}{\delta\mathcal{G}_{0}(0,Z)}\mathcal{G}_{0}(\theta',\theta,Z)\right)\right)_{\theta=\theta'}$$

$$= \left(\omega^{-1}\left(\mathcal{G}_{0}(0,Z)\right)+\mathcal{G}_{0}(0,Z)\frac{\delta\left[\omega^{-1}\left(\bar{J},Z,\mathcal{G}_{0}\right)\right]}{\delta\mathcal{G}_{0}(0,Z)}\right)\nabla_{\theta}+\frac{\delta\left[\omega^{-1}\left(\bar{J},Z,\mathcal{G}_{0}\right)\right]}{\delta\mathcal{G}_{0}(0,Z)}\left(\nabla_{\theta}\mathcal{G}_{0}(\theta',\theta,Z)\right)_{\theta=\theta'}$$

As computed before, $\omega^{-1}\left(\bar{J},Z,\mathcal{G}_{0}\right)$ satisfies:

$$\omega^{-1}\left(\bar{J},Z,\mathcal{G}_{0}\right)=G\left(\bar{J}+\frac{\kappa}{N}\int T(Z,Z_{1})W\left(\frac{\omega\left(\bar{J},Z,\mathcal{G}_{0}\right)}{\omega\left(\bar{J},Z_{1},\mathcal{G}_{0}\right)}\right)\frac{\omega\left(\bar{J},Z_{1},\mathcal{G}_{0}\right)\mathcal{G}_{0}(Z_{1},0)\,dZ_{1}}{\omega\left(\bar{J},Z,\mathcal{G}_{0}\right)}\right)$$

3.2 General formula for the effective action

The perturbative corrections for the effective action are found by adding the 1PI graphs with $2n$ external points with $n\geqslant 2$. The corrections are ordered by the number of vertices involved in them.

3.2.1 First and second order corrections

Using that, for $n\geqslant 2$:

$$\left[\frac{\delta^{n}\left[\int\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\omega^{-1}(J,\theta,Z)\Psi(\theta,Z)\,dZd\theta+V(\Psi)\right]}{\prod_{i=1}^{n}\delta\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}}\right]_{\substack{|\Psi(\theta,Z)|^{2}\\=\mathcal{G}_{0}(0,Z)}}=\left[\frac{\delta^{n}\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{i=1}^{n}\delta\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}}\right]_{\substack{|\Psi(\theta,Z)|^{2}\\=\mathcal{G}_{0}(0,Z)}}$$

the lowest order expansion of 1PI graphs, consists of graphs with n horizontal propagators, connected by one vertex of valence n. The sum of these contributions becomes:

$$\Gamma_{0}\left(\Psi^{\dagger},\Psi\right)+\sum_{n=2}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\Psi^{\dagger}\left(\theta_{i}^{(i)},Z_{i}\right)\left[\frac{\delta^{n}\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{i=1}^{n}\delta\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}}\right]_{\substack{|\Psi(\theta,Z)|^{2}\\=\mathcal{G}_{0}(0,Z)}}\Psi\left(\theta_{i}^{(i)},Z_{i}\right)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\Psi^{\dagger}\left(\theta_{i}^{(i)},Z_{i}\right)\left[\frac{\delta^{n}\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{i=1}^{n}\delta\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}}\right]_{\substack{|\Psi(\theta,Z)|^{2}\\=\mathcal{G}_{0}(0,Z)}}\Psi\left(\theta_{i}^{(i)},Z_{i}\right)$$

so that, up to the field independent term $\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]_{|\Psi(\theta,Z)|^{2}=\mathcal{G}_{0}(0,Z)}$, we have the effective action at the first order in vertices:

$$\begin{aligned}\Gamma_{0}\left(\Psi^{\dagger},\Psi\right)+\Gamma_{1}\left(\Psi^{\dagger},\Psi\right) &= S_{cl}\left(\mathcal{G}_{0}(0,Z)+|\Psi|^{2}\right)\\ &\equiv -\frac{1}{2}\int\left(\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(|\Psi(\theta,Z)|^{2}\right)\right)\right)\left(\mathcal{G}_{0}(\theta',\theta,Z)+\Psi^{\dagger}(\theta',Z)\Psi(\theta,Z)\right)\right)_{\theta'=\theta}\\ &\quad +\alpha\int\left(\mathcal{G}_{0}(0,Z_{i})+\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}\right)+\sum_{n\geqslant 2}V_{n}\left(\mathcal{G}_{0}(0,Z_{i})+\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}\right)\end{aligned}$$

To find the second order in vertices-number corrections, we need to define the notion of multiple point. Consider, for $i = 1, ..., n$, the set $\left(\left(\theta_f^{(i)}, Z_i\right), \left(\theta_i^{(i)}, Z_i\right)\right)$ of n initial points and n final points and any graph connecting the initial and final points . The index $i \in \{1, ..., n\}$ labels a multiple point of valence k of the graph, if the line connecting $\left(\theta_f^{(i)}, Z_i\right)$ and $\left(\theta_i^{(i)}, Z_i\right)$ is reached by k legs of the vertices defining the graph. A multiple point of valence 2 is a double point and a multiple point of valence l is also refered as a l-multiple point.

At the second order in products of vertices the sum of 1PI graphs is:

$$\int \sum_{n\geqslant 2} \frac{1}{n!} \sum_{\substack{l_1=2,l_2=2\\ l_1+l_2\geqslant n+2}}^{n} \sum_{\substack{\{k_1,...,k_{l_1}\}\cup\{k'_1,...,k'_{l_2}\}\\ =\{1,...,n\}}} \left(\prod_{k_d\in D} \Psi^\dagger\left(\theta_f^{(k_d)}, Z_i\right) \frac{\exp\left(-\Lambda_1\left(\theta_f^{(k_d)} - \theta_i^{(k_d)}\right)\right)}{\Lambda^{l_1+l_2}}\right) \prod_{i=1, i\notin D}^{n} \Psi^\dagger\left(\theta_i^{(i)}, Z_i\right)$$

$$\times \left(\left[\frac{\delta^{l_1}\left[S_{cl}\left(\Psi^\dagger, \Psi\right)\right]}{\prod_{i=1}^{l_1} \delta\left|\Psi\left(\theta^{(k_i)}, Z_{k_i}\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\ =\mathcal{G}_0(0,Z)}}\right)\left(\left[\frac{\delta^{l_2}\left[S_{cl}\left(\Psi^\dagger, \Psi\right)\right]}{\prod_{i=1}^{l_2} \delta\left|\Psi\left(\theta^{(k_i)}, Z_{k_i}\right)\right|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}\right)\prod_{i=1}^{n}\Psi\left(\theta_i^{(i)}, Z_i\right) \tag{202}$$

with D the set of double points of $\{1, ..., n\}$, that is $\{k_1, ..., k_{l_1}\} \cap \{k'_1, ..., k'_{l_2}\}$.

3.2.2 Including higher order corrections

Including all contributions is straightforward and generalizes (202). The sum of these contributions are:

$$S_{cl}\left(\mathcal{G}_0(0, Z) + |\Psi|^2\right)$$
$$+ \int \sum_{n\geqslant 2, p\geqslant 2} \frac{1}{n!} \sum_{\substack{l_1+...l_p\geqslant n+2\\ 2\leqslant l_m\leqslant n}} \sum_{\substack{\cup_{j=1...p}\{k_1,...,k_{l_j}\}\\ =\{1,...,n\}}} \left(\prod_{k_d\in D} \Psi^\dagger\left(\theta_f^{(k_d)}, Z_{k_d}\right) \prod_{i=1, i\notin D}^{n} \Psi^\dagger\left(\theta_i^{(i)}, Z_i\right)\right)$$
$$\times \frac{\exp\left(-\Lambda_1\left(\theta_f^{(k_d)} - \theta_i^{(k_d)}\right)\right)}{\Lambda^{l_1+l_2}} \left(\prod_{j=1}^{p}\left[\frac{\delta^{l_j}\left[S_{cl}\left(\Psi^\dagger, \Psi\right)\right]}{\prod_{i=1}^{l_j} \delta\left|\Psi\left(\theta^{(k_i)}, Z_{k_i}\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\ =\mathcal{G}_0(0,Z)}}\right)\prod_{i=1}^{n}\Psi\left(\theta_i^{(i)}, Z_i\right)$$

where D is the subset of multiple points of $\{1, ..., n\}$, that is the elements belonging at least to two distinct sets $\{k_1, ..., k_{l_j}\}$. The sum is constrained to 1PI graphs.

The sum can be regrouped in a different way. The graphs can be gathered in classes with respects to the number of multiple points and the legs ending at these points.

To do so, we split the multiple points on the the line l_i connecting $\left(\theta_f^{(i)}, Z_i\right)$ and $\left(\theta_i^{(i)}, Z_i\right)$ into multiple individual points by cutting the segments between any two vertices endpoints. The associated resulting graph belongs to the reduced class. This is the class of graphs in which the vertices with n legs are connected to n points once, and two different vertices are connected to different points.

The contribution of a graph in the reduced class with p vertices of valence l_j, $j = 1, ..., p$ is a product:

$$\frac{1}{\prod(\natural l_j)!}\prod_{j=1}^{p}\left(\prod_{i=1}^{l_j}\Psi^\dagger\left(\theta^{(k_i)}, Z_{k_i}\right)\left[\frac{1}{l_j!}\frac{\delta^{l_j}\left[S_{cl}\left(\Psi^\dagger, \Psi\right)\right]}{\prod_{i=1}^{l_j} \delta\left|\Psi\left(\theta^{(k_i)}, Z_{k_i}\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\ =\mathcal{G}_0(0,Z)}}\prod_{i=1}^{l_j}\Psi\left(\theta^{(k_i)}, Z_{k_i}\right)\right)$$

of independent terms, where $(\natural l_j)$ is the number of vertices of identical valence and the product runs other the set of valences present in the graph. The graphs in the reduced class can be summed to produce global contributions of the form:

$$\frac{1}{p!}\sum\prod_{j=1}^{p}\left(\prod_{i=1}^{l_j}\Psi^{\dagger}\left(\theta^{(k_i)},Z_{k_i}\right)\left[\sum_{l_j}\frac{1}{l_j!}\frac{\delta^{l_j}\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{i=1}^{l_j}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\prod_{i=1}^{l_j}\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right)$$

$$= \frac{1}{p!}\prod_{j=1}^{p}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$$

where:

$$\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\equiv S_{cl}\left(\mathcal{G}_0\left(0,Z\right)+|\Psi|^2\right)$$

The factor $\frac{1}{p!}$ arises from the graph expansion of $\prod_{j=1}^{p}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$. Actually, expanding the products yields a multiplicity of the products of contribution which is $\frac{p!}{\prod(\natural l_j)!}$. Dividing by $p!$ thus restores the factor $\frac{1}{\prod(\natural l_j)!}$.

Reintroducing the multiple points to compute the graphs in a certain class amounts to differentiate the factors $\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$ with respect to $|\Psi|^2$and to introduce products of fields corresponding to these multiple points.

Actually, let us consider the graphs with p vertices and $n=\sum_{i\geqslant 1}l^i$ external points with l^i points of multiplicity i. We have $l^i=\sum_{j=1}^{p}l_j^i$, where l_j^i is the number of legs of vertex j connected to a point of multiplicity i. The factor associated to the repartition of the simple and multiple points times the global factor $\frac{1}{n!}$ is thus:

$$\frac{1}{n!}C_n^{l^1}C_{n-l^1}^{l^2}...=\prod_i\frac{1}{(l^i)!}$$

The attribution of the simple points yields the factors $\prod\frac{1}{l_j^1!}$. A factor $\frac{1}{\prod(\natural\sum_{i\geqslant 1}l_j^i)!}$ is associated for identical vertices. The multiple points are then connected to the vertices in all possible manners compatible with the l_j^i.

Then, starting with a factor:

$$\frac{1}{\prod(\natural l_j)!}\prod_{j=1}^{p}\left(\prod_{i=1}^{l_j}\Psi^{\dagger}\left(\theta^{(k_i)},Z_{k_i}\right)\left[\frac{1}{l_j!}\frac{\delta^{l_j}\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{i=1}^{l_j}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\prod_{i=1}^{l_j}\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right)$$

and applying:

$$\frac{1}{2}\int\frac{\delta^2}{\delta\left|\Psi\left(\theta^{(1)},Z\right)\right|^2\delta\left|\Psi\left(\theta^{(2)},Z\right)\right|^2}d\theta^{(1)}d\theta^{(2)}$$

yields two factors of the form:

$$\left(\prod_{i=1}^{l_j-1}\Psi^{\dagger}\left(\theta^{(k_i)},Z_{k_i}\right)\left[\frac{1}{(l_j-1)!}\frac{\delta^{l_j}\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\Lambda\prod_{i=1}^{l_j-1}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2\delta\left|\Psi\left(\theta^{(1,2)},Z\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\prod_{i=1}^{l_j-1}\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right)$$

or one factor:

$$\left(\prod_{i=1}^{l_j-2}\Psi^{\dagger}\left(\theta^{(k_i)},Z_{k_i}\right)\left[\frac{1}{(l_j-2)!}\frac{\delta^{l_j}\left[S_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\Lambda\prod_{i=1}^{l_j-2}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2\delta\left|\Psi\left(\theta^{(1)},Z\right)\right|^2\delta\left|\Psi\left(\theta^{(2)},Z\right)\right|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}\prod_{i=1}^{l_j-2}\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right)$$

This corresponds to the introduction of a double point, with the required factors for simple points, the overall factor $\frac{1}{\prod(\natural l_j)!}$ corresponding to $\frac{1}{\prod(\natural\sum_{i\geqslant 1}l_j^i)!}$ with $l_j^1=l_j$ except for the two factors case where $l_j^1=l_j-1$ and $l_j^2=1$, or for the single factor $l_j^1=l_j-2$ and $l_j^2=2$.

To account for the global factors $\frac{1}{l^i!}$ and the propagators associated to the multiple points, the operator corresponding to the introduction of k double points is then:

$$\frac{1}{k!}\left(\frac{1}{2}\int\Psi^{\dagger}\left(\theta_f,Z\right)\frac{\exp\left(-\Lambda_1\left(\theta_f-\theta_i\right)\right)}{\Lambda^2}\Psi\left(\theta_i,Z\right)\int\frac{\delta^2}{\delta\left|\Psi\left(\theta^{(1)},Z\right)\right|^2\delta\left|\Psi\left(\theta^{(2)},Z\right)\right|^2}d\theta^{(1)}d\theta^{(2)}\right)^k$$

More generally, the introduction of l-multiple points are generated by operators:

$$\frac{1}{k!}\left(\frac{1}{l!}\int\Psi^{\dagger}\left(\theta_f,Z\right)\frac{\exp\left(-\Lambda_1\left(\theta_f-\theta_i\right)\right)}{\Lambda^l}\Psi\left(\theta_i,Z\right)\int\frac{\delta^2}{\prod_{s=1}^{l}\delta\left|\Psi\left(\theta^{(s)},Z\right)\right|^2}\prod_{s=1}^{l}d\theta^{(s)}\right)^k$$

So that the whole series generating the multiple points is:

$$\begin{aligned}
&\Gamma\left(\Psi^{\dagger},\Psi\right) \qquad (203)\\
=\ &\exp\left(\sum_{l\geqslant 2}\frac{1}{l!}\int\Psi^{\dagger}\left(\theta_f,Z\right)\frac{\exp\left(-\Lambda_1\left(\theta_f-\theta_i\right)\right)}{\Lambda^l}\Psi\left(\theta_i,Z\right)\int\frac{\delta^2}{\Lambda\prod_{s=1}^{l}\delta\left|\Psi\left(\theta^{(s)},Z\right)\right|^2}\prod_{s=1}^{l}d\theta^{(s)}\right)\sum\frac{1}{p!}\prod_{j=1}^{p}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\\
=\ &\exp\left(\sum_{l\geqslant 2}\frac{1}{l!}\int\Psi^{\dagger}\left(\theta_f,Z\right)\frac{\exp\left(-\Lambda_1\left(\theta_f-\theta_i\right)\right)}{\Lambda^l}\Psi\left(\theta_i,Z\right)\int\frac{\delta^2}{\prod_{s=1}^{l}\delta\left|\Psi\left(\theta^{(s)},Z\right)\right|^2}\prod_{s=1}^{l}d\theta^{(s)}\right)\exp\left(\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right)
\end{aligned}$$

It is understood that only the 1PI graphs are kept in the series expansion.

The expansion can also be written in expanded form and yields the effective action $\Gamma\left(\Psi^{\dagger},\Psi\right)$:

$$\begin{aligned}
\Gamma\left(\Psi^{\dagger},\Psi\right)&=\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right) \qquad (204)\\
&\times\prod_{i=1}^{m}\left[\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right]\\
&\times\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{m!\prod_k\left(\sharp_k\right)!\Lambda^{\sum_i p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)
\end{aligned}$$

with:

$$\sharp_{k,j,m}\left(\left(p_l^i\right)\right)=\sum_{l=1}^{j}\delta_{k,\sum_{i=1}^{m}p_l^i} \tag{205}$$

and where:

$$\theta_i^{(l)}<\theta^{\left(k_l^i\right)}<\theta_f^{(l)}$$

The notation $\left(p_l^i\right)$ in (205) stands for the dependency of $\sharp_{k,j,m}$ in the whole collection of indices $\left(p_l^i\right)$, $i=1...m$ and $l=1...j$.

The sums over the indices in (204) represent the sum over the different class of graphs with respect to the multiple points. Actually, to each of the m copies of $\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$ we associate a point P_i and to each Z_l we associate a line L_l. For each derivative of the copy with respect to $\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2$ we draw a segment from P_i to L_l. The number of segments between P_i and L_l is equal to p_l^i. This produces a graph with multiple points and the corresponding expression computes the sum of graphs in the class of this multiple-points graph. The sum over the indices are constrained to produce 1PI graphs.

In the local approximation, we replace $\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{\Lambda^2}$ by $\frac{\delta\left(\theta_f^{(l)}-\theta_i^{(l)}\right)}{\Lambda_1\Lambda^2}$, and as a consequence the effective action writes:

$$\begin{aligned}\Gamma\left(\Psi^{\dagger},\Psi\right) &= \hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\\ &+\sum_{j\geqslant 2}\int\left[\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\sum_{\substack{m\geqslant 1,\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}\prod_{i=1}^{m}\frac{\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right]}{m!\prod_k\left(\sharp_{k,j,m}\left(\left(p_l^i\right)\right)\right)!\Lambda_1^j\Lambda^{\sum_{i,l}p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta^{(l)},Z_l\right)\right)\right]\end{aligned}$$

Note that, for $\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_l p_l^i!\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}<<1$ when $\sum_l p_l^i$ increases, the dominant part of graphs for $j\geqslant 2$ is for $m=j$ and $\sum_l p_l^i=2$. In this case the effective action rewrites:

$$\begin{aligned}\Gamma\left(\Psi^{\dagger},\Psi\right)&=\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{j\geqslant 2}\sum_{\substack{\cup\left(l_1^i,l_2^i\right)_{i=1,\dots j}\\ =(1,\dots,j)^2}}\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\\ &\times\prod_{i=1}^{j}\left[\int_{\left[\theta_i^{\left(l_1^i\right)},\theta_f^{\left(l_1^i\right)}\right]\times\left[\theta_i^{\left(l_2^i\right)},\theta_f^{\left(l_2^i\right)}\right]}\frac{\delta^2\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\delta\left|\Psi\left(\theta^{\left(l_1^i\right)},Z_{l_1^i}\right)\right|^2\delta\left|\Psi\left(\theta^{\left(l_2^i\right)},Z_{l_2^i}\right)\right|^2}d\theta^{\left(l_1^i\right)}d\theta^{\left(l_2^i\right)}\right]\\ &\times\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{\Lambda^2}\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)\end{aligned}$$

and it the local approximation, we have:

$$\Gamma\left(\Psi^{\dagger},\Psi\right)=\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$$
$$+\sum_{j\geqslant 2}\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_{f}^{(l)},Z_{l}\right)\right)\prod_{i=1}^{j}\left[\int_{\left[\theta_{i}^{(i)},\theta_{f}^{(i+1)}\right]\times\left[\theta_{i}^{(i)},\theta_{f}^{(i+1)}\right]}\frac{\delta^{2}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\delta\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}\delta\left|\Psi\left(\theta^{(i+1)},Z_{i+1}\right)\right|^{2}}d\theta^{(i)}d\theta^{(i+1)}\right]$$
$$\times\frac{\exp\left(-\Lambda_{1}\left(\theta_{f}^{(l)}-\theta_{i}^{(l)}\right)\right)}{\Lambda^{2}}\left(\prod_{l=1}^{j}\Psi\left(\theta_{i}^{(l)},Z_{l}\right)\right)$$

with the convention that $i+1\equiv 1$ for $i=j$.

3.3 Alternative form for the effective action

For later purposes, (204) can be written in developped form. To dos so, we first give an expanded form for $\prod_{i=1}^{m}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$ arising in (204).

3.3.1 Expanded form for $\prod_{i=1}^{m}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$

We start with:

$$\begin{aligned}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right) &= -\frac{1}{2}\int\left(\left(\nabla_{\theta}\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(\left|\Psi\left(\theta,Z\right)\right|^{2}\right)\right)\left(\mathcal{G}_{0}\left(\theta',\theta,Z\right)+\Psi^{\dagger}\left(\theta',Z\right)\Psi\left(\theta,Z\right)\right)\right)_{\theta'=\theta}\\ &\quad+\alpha\int\left(\mathcal{G}_{0}\left(0,Z_{i}\right)+\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}\right)+\sum_{n\geqslant 2}V_{n}\left(\mathcal{G}_{0}\left(0,Z_{i}\right)+\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}\right)\end{aligned}$$

Given the form of $V_{n}\left(\mathcal{G}_{0}\left(0,Z_{i}\right)+\left|\Psi\left(\theta^{(i)},Z_{i}\right)\right|^{2}\right)$ and given that:

$$\begin{aligned}&-\frac{1}{2}\int\left(\left(\nabla_{\theta}\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(\left|\Psi\left(\theta,Z\right)\right|^{2}\right)\right)\left(\mathcal{G}_{0}\left(\theta',\theta,Z\right)+\Psi^{\dagger}\left(\theta',Z\right)\Psi\left(\theta,Z\right)\right)\right)_{\theta'=\theta}\\ \simeq&-\frac{1}{2}\int\Psi^{\dagger}\left(\theta,Z\right)\left(\nabla_{\theta}\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(\left|\Psi\left(\theta,Z\right)\right|^{2}\right)\right)\Psi\left(\theta,Z\right)\end{aligned}$$

we can rewrite $\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$, up to the constant $\alpha\int\mathcal{G}_{0}\left(0,Z_{i}\right)$:

$$\begin{aligned}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right) &\simeq \int\Psi^{\dagger}\left(\theta,Z\right)\left(-\frac{1}{2}\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(\left|\Psi\left(\theta,Z\right)\right|^{2}\right)\right)\right)+\alpha+U\left(\left|\Psi\left(\theta,Z\right)\right|^{2}\right)\right)\Psi\left(\theta,Z\right)\\ &\equiv \int\Psi^{\dagger}\left(\theta,Z\right)L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\Psi\left(\theta,Z\right)\end{aligned}\qquad(206)$$

where $U\left(\int\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\right)$ is obtained by the series expansion of $U_{0}\left(\mathcal{G}_{0}\left(0,Z_{i}\right)+\int\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\right)$ and collecting its terms of degree 2 and higher in fields.

Then, in (204), The product of m copies of $\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$ can be reordered as:

$$\prod_{i=1}^{m}\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)=m!\int_{\theta_{1}<\ldots<\theta_{m}}\Psi^{\dagger}\left(\theta_{i},Z_{i}\right)L\left(\Psi^{\dagger}\left(\theta_{i},Z_{i}\right),\Psi\left(\theta_{i},Z_{i}\right)\right)\Psi\left(\theta_{i},Z_{i}\right)$$

and (204) becomes:

$$\Gamma\left(\Psi^{\dagger},\Psi\right)=\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_{l}^{i}\right)_{m\times j}\\ \sum_{i}p_{l}^{i}\geqslant 2}}\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_{f}^{(l)},Z_{l}\right)\right) \tag{207}$$

$$\times\int_{\theta_{1}<\ldots<\theta_{m}}\prod_{i=1}^{m}\left[\int_{\prod_{l=1}^{j}\left[\theta_{i}^{(l)},\theta_{f}^{(l)}\right]^{p_{l}^{i}}}\frac{\delta^{\sum_{l}p_{l}^{i}}\left[\Psi^{\dagger}\left(\theta_{i},Z_{i}\right)L\left(\Psi^{\dagger}\left(\theta_{i},Z_{i}\right),\Psi\left(\theta_{i},Z_{i}\right)\right)\Psi\left(\theta_{i},Z_{i}\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{i}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{i}}d\theta^{\left(k_{l}^{i}\right)}\right]d\left(\theta_{i},Z_{i}\right)$$

$$\times\frac{\exp\left(-\Lambda_{1}\left(\theta_{f}^{(l)}-\theta_{i}^{(l)}\right)\right)}{\prod_{k}\left(\sharp_{k,j,m}\left(\left(p_{l}^{i}\right)\right)\right)!\Lambda^{\sum_{i}p_{l}^{i}}}\left(\prod_{l=1}^{j}\Psi\left(\theta_{i}^{(l)},Z_{l}\right)\right)$$

3.3.2 Computation of the factor for $i=m$ in (207)

To compute:

$$\frac{\delta^{\sum_{l}p_{l}^{m}}\left[\Psi^{\dagger}\left(\theta_{m},Z_{m}\right)L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\Psi\left(\theta_{m},Z_{m}\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}} \tag{208}$$

in (207), we decompose this term as a sum:

$$\Psi^{\dagger}\left(\theta_{m},Z_{m}\right)\frac{\delta^{\sum_{l}p_{l}^{m}}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}\Psi\left(\theta_{m},Z_{m}\right) \tag{209}$$

$$+\sum_{\substack{p_{l}^{m\prime}\leqslant p_{l}^{m}\\ \sum_{l}p_{l}^{m\prime}=\sum_{l}p_{l}^{m}-1}}\frac{\delta^{\sum_{l}p_{l}^{m}-1}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m\prime}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}$$

Each configuration $\left(p_{l}^{m\prime}\right)$ is reached by j configuration $\left(p_{l}^{m}\right)$. We perform a change of variable $p_{l}^{m\prime}\rightarrow p_{l}^{m}$, and the factor $\frac{1}{\prod_{k}\left(\sharp_{j,m;k}\right)!}$ for each configuration has to be replaced for each of the j configurations it is issued from.

Now, we consider the second term in (209) multiplied by $\frac{1}{\prod_{k}\left(\sharp_{k,j,m}\left(\left(p_{l}^{i}\right)\right)\right)!}=\frac{1}{\sharp_{j,m}\left(\left(p_{l}^{i}\right)\right)}$:

$$\frac{1}{\prod_{k}\left(\sharp_{j,m,k}\right)!}\frac{\delta^{\sum_{l}p_{l}^{m}-1}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m\prime}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}$$

$$=\sum_{p=1}^{j}\frac{1}{\prod_{k}\left(\sum_{l=1}^{j}\delta_{k,\sum_{i=1}^{m}p_{l}^{i}}\right)!}\frac{\delta^{\sum_{l}p_{l}^{m}-1}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m\prime}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}$$

In the computation of (207), this term can be replaced by:

$$\left(\sum_{p=1}^{j}\frac{1}{\prod_{k}\left(\sum_{l}\delta_{k,\sum_{i}p_{l}^{i}+\delta_{l,p}}\right)!}\right)\frac{\delta^{\sum_{l}p_{l}^{m}}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}$$

$$=\frac{\delta^{\sum_{l}p_{l}^{m}}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\bar{\sharp}_{j,m}\left(\left(p_{l}^{i}\right)\right)\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}$$

where we define:

$$\frac{1}{\bar{\sharp}_{j,m}\left(\left(p_{l}^{i}\right)\right)}=\frac{1}{\prod_{k}\left(\bar{\sharp}_{j,m,k}\left(\left(p_{l}^{i}\right)\right)\right)!}=\sum_{p=1}^{j}\frac{1}{\prod_{k}\left(\sum_{l=1}^{j}\delta_{k,\sum_{i=1}^{m}p_{l}^{i}+\delta_{l,p}}\right)!}\tag{210}$$

As a consequence (208) can be replaced in (207) by:

$$\frac{\delta^{\sum_{l}p_{l}^{m}}\left[\Psi^{\dagger}\left(\theta_{m},Z_{m}\right)L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\Psi\left(\theta_{m},Z_{m}\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}\tag{211}$$

$$\rightarrow\quad\Psi^{\dagger}\left(\theta_{m},Z_{m}\right)\frac{\delta^{\sum_{l}p_{l}^{m}}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}\Psi\left(\theta_{m},Z_{m}\right)$$

$$+\frac{\sharp_{j,m}\left(\left(p_{l}^{i}\right)\right)}{\bar{\sharp}_{j,m}\left(\left(p_{l}^{i}\right)\right)}\frac{\delta^{\sum_{l}p_{l}^{m}}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}$$

where:

$$\sharp_{j,m}\left(\left(p_{l}^{i}\right)\right)=\prod_{k}\left(\sharp_{k,j,m}\left(\left(p_{l}^{i}\right)\right)\right)!\tag{212}$$

and $\bar{\sharp}_{j,m}\left(\left(p_{l}^{i}\right)\right)$ is defined by (210).

3.3.3 Reintroducing (211) in (207) and local approximation

The first contribution in the RHS of (211):

$$\Psi^{\dagger}\left(\theta_{m},Z_{m}\right)\frac{\delta^{\sum_{l}p_{l}^{m}}\left[L\left(\Psi^{\dagger}\left(\theta_{m},Z_{m}\right),\Psi\left(\theta_{m},Z_{m}\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{m}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}\Psi\left(\theta_{m},Z_{m}\right)\tag{213}$$

once reintroduced in the effective action (207), yields:

$$\int_{\theta_1<\ldots<\theta_m}\Psi^{\dagger}(\theta_m,Z_m)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}(\theta_m,Z_m),\Psi(\theta_m,Z_m)\right)\right]}{\prod\limits_{l=1}^{j}\prod\limits_{k_l^i=1}^{p_l^m}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\Psi(\theta_m,Z_m)\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right) \tag{214}$$

$$\times\prod_{i=1}^{m-1}\left[\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\Psi^{\dagger}(\theta_i,Z_i)L\left(\Psi^{\dagger}(\theta_i,Z_i),\Psi(\theta_i,Z_i)\right)\Psi(\theta_i,Z_i)\right]}{\prod\limits_{l=1}^{j}\prod\limits_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right]d(\theta_i,Z_i)$$

$$\times\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{\prod\limits_k\left(\sharp_{j,k}\left(\left(p_l^i\right)\right)\right)!\Lambda^{\sum_i p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)$$

$$=\int_{\theta_m}\Psi^{\dagger}(\theta_m,Z_m)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}(\theta_m,Z_m),\Psi(\theta_m,Z_m)\right)\right]}{\prod\limits_{l=1}^{j}\prod\limits_{k_l^i=1}^{p_l^m}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\Psi(\theta_m,Z_m)$$

$$\times\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m-1}\left[\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta_m}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod\limits_{l=1}^{j}\prod\limits_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right]$$

$$\times\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{(m-1)!\prod\limits_k\left(\sharp_{j,k}\left(\left(p_l^i\right)\right)\right)!\Lambda^{\sum_i p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)$$

with:

$$\hat{S}_{cl,\theta_m}\left(\Psi^{\dagger},\Psi\right)=\int^{\theta_m}\Psi^{\dagger}(\theta,Z)L\left(\Psi^{\dagger}(\theta,Z),\Psi(\theta,Z)\right)\Psi(\theta,Z)$$

The second contribution in (211) is constrained by that, among the fields $\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)$ one of them is set to $\Psi^{\dagger}(\theta_m,Z_m)$. This leads to the following contribution to (207):

$$\int_{\theta_m}\Psi^{\dagger}(\theta_m,Z_m)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}(\theta_m,Z_m),\Psi(\theta_m,Z_m)\right)\right]}{\prod\limits_{l=1}^{j-1}\prod\limits_{k_l^i=1}^{p_l^m}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\Psi\left(\theta_i^{(m)},Z_m\right) \tag{215}$$

$$\times\int\left(\prod_{l=1}^{j-1}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m-1}\left[\int_{\prod_{l=1}^{j-1}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta_m}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod\limits_{l=1}^{j}\prod\limits_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j-1}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right]$$

$$\times\frac{\sharp_{j,m}\left(\left(p_l^i\right)\right)}{\sharp_{j,m}\left(\left(p_l^i\right)\right)}\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{(m-1)!\Lambda^{\sum_i p_l^i}}\left(\prod_{l=1}^{j-1}\Psi\left(\theta_i^{(l)},Z_l\right)\right)$$

and the product of derivatives can be decomposed by isolating those corresponding to Z_m:

$$\int_{\prod_{l=1}^{j-1}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}} \frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta_m}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j-1}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}$$

$$= \int_{\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p^i}} \frac{\delta^{p^i}}{\prod_{k_l^i=1}^{p^i}\delta\left|\Psi\left(\theta^{(k^i)},Z_m\right)\right|^2}d\theta^{\left(k^i\right)}\int_{\prod_{l=1}^{j-1}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}} \frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta_m}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j-1}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j-1}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}$$

Gathering the contributions (214) and (215) yields $\Gamma\left(\Psi^{\dagger},\Psi\right)$:

$$\begin{aligned}
\Gamma\left(\Psi^{\dagger},\Psi\right) &= \hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right) \\
&+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_l p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}a_{j,m}\left(\theta,\theta_i\right)\Psi\left(\theta_i,Z\right) \\
&\times\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left[\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}\right] \\
&\times\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{m!\prod_k\left(\sharp_{j,k}\left(\left(p_l^i\right)\right)\right)!\Lambda^{\sum_i p_l^i}}\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)
\end{aligned} \tag{216}$$

with

$$a_1\left(\theta,\theta_i\right)=\frac{\exp\left(-\Lambda_1\left(\theta-\theta_i^{(l)}\right)\right)}{\Lambda^{\sum_i p_l^i}}\prod_{i=1}^{m}\int_{\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p^i}}\frac{\delta^{p^i}}{\prod_{k_l^i=1}^{p^i}\delta\left|\Psi\left(\theta^{(k^i)},Z_m\right)\right|^2}d\theta^{\left(k^i\right)}$$

and

$$a_{j,m}\left(\theta,\theta_i\right)=\delta\left(\theta-\theta_i\right)+\frac{\sharp_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)}{\bar{\sharp}_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)}\frac{\exp\left(-\Lambda_1\left(\theta-\theta_i^{(l)}\right)\right)}{\Lambda^{\sum_i p_l^i}}\prod_{i=1}^{m}\int_{\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p^i}}\frac{\delta^{p^i}}{\prod_{k_l^i=1}^{p^i}\delta\left|\Psi\left(\theta^{(k^i)},Z_m\right)\right|^2}d\theta^{\left(k^i\right)}$$

for $j>1$. The derivatives $i=1,,,,m$ implicitly act independently on each factor:

$$\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^{p_l^i}}\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}d\theta^{\left(k_l^i\right)}$$

in (216).

The notations $\sharp_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)$ and $\bar{\sharp}_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)$ are defined by (210) and (212) in which the multi-indices $\left(p_l^i\right)_{l=1,\ldots l}^{i=1,\ldots m}$ are replaced by the collection for $m+1$ and $j+1$ obtained by gathering $\left(p_l\right)_{i=1\ldots m}$ and $\left(p_l^i\right)_{l=1,\ldots l}^{i=1,\ldots m}$.

In the local approximation, equation (216) simplifies and writes:

$$\Gamma\left(\Psi^{\dagger},\Psi\right)=\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right) \tag{217}$$
$$+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_l p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)$$
$$\times\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta_l,Z_l\right)\right|^2}\right]\frac{\left(\prod_{l=1}^{j}\Psi\left(\theta^{(l)},Z_l\right)\right)}{m!\prod_k\left(\sharp_{j,k}\right)!\Lambda^{\sum_i p_l^i}}\prod_{l=1}^{j}dZ_l d\theta_l$$

with:

$$a_{1,m}=1$$

and

$$a_{j,m}=1+\frac{\sharp_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)}{\sharp_{j+1,m+1}\left(\left(p_l\right),\left(p_l^i\right)\right)}$$

for $j>1$.

3.3.4 Recursive expansion of (217)

The procedure that led to rewrite (213), can be applied in (217), to the terms:

$$\frac{1}{m!}\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta_l,Z_l\right)\right|^2}\right] \tag{218}$$

Expression (218) can be expanded recursively. As we did above, expression (218) is replaced by an integration over the restricted domain $\theta_1<\theta_m,...,\theta_{m-1}<\theta_m<\theta$:

$$\int_{\theta_1<\theta_m,...,\theta_{m-1}<\theta_m<\theta}\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\left[\Psi^{\dagger}\left(\theta_i,Z_i\right)L\left(\Psi^{\dagger}\left(\theta_i,Z_i\right),\Psi\left(\theta_i,Z_i\right)\right)\Psi\left(\theta_i,Z_i\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta_l,Z_l\right)\right|^2}\right]d\left(\theta_i,Z_i\right)$$

and formula (211) applies to replace:

$$\frac{\delta^{\sum_l p_l^m}\left[\Psi^{\dagger}\left(\theta_m,Z_m\right)L\left(\Psi^{\dagger}\left(\theta_m,Z_m\right),\Psi\left(\theta_m,Z_m\right)\right)\Psi\left(\theta_m,Z_m\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta_l,Z_l\right)\right|^2}$$

by:

$$\Psi^{\dagger}\left(\theta_m,Z_m\right)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}\left(\theta_m,Z_m\right),\Psi\left(\theta_m,Z_m\right)\right)\right]}{\prod_{k_l^i=1}^{p_l^m}\delta^{p_l^m}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta_m,Z_m\right)+\frac{\sharp_{j,m}\left(\left(p_l^i\right)\right)}{\sharp_{j,m}\left(\left(p_l^i\right)\right)}\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}\left(\theta_m,Z_m\right),\Psi\left(\theta_m,Z_m\right)\right)\right]}{\prod_{k_l^i=1}^{p_l^m}\delta^{p_l^m}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2} \tag{219}$$

Inserting the first term of expression (219) in (217) yields the contribution:

$$\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_l p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right) \tag{220}$$

$$\times\int\Psi^{\dagger}\left(\theta_m,Z_m\right)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}\left(\theta_m,Z_m\right),\Psi\left(\theta_m,Z_m\right)\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^m}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta_m,Z_m\right)$$

$$\times\int a_{j,m}\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\prod_{i=1}^{m-1}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right]\frac{\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)}{(m-1)!\prod_k\left(\sharp_{j,k}\left(\left(p_l^i\right)\right)\right)!\Lambda^{\sum_i p_l^i}}\prod_{l=1}^{j}dZ_l d\theta^{(l)}$$

The second contribution of (219), after insertion in (216) yields:

$$\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_l p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right) \tag{221}$$

$$\times\int\Psi^{\dagger}\left(\theta_m,Z_m\right)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}\left(\theta_m,Z_m\right),\Psi\left(\theta_m,Z_m\right)\right)\right]}{\prod_{l=1}^{j-1}\delta^{p_l^m}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta_m,Z_m\right)$$

$$\times\int\left(\prod_{l=1}^{j-1}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\prod_{i=1}^{m-1}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta_m}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j-1}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right]\times\frac{\sharp_{j,m}\left(\left(p_l^m\right),\left(p_l^i\right)\right)}{\sharp_{j,m}\left(\left(p_l^m\right),\left(p_l^i\right)\right)}a_{j,m}\frac{\left(\prod_{l=1}^{j-1}\Psi\left(\theta^{(l)},Z_l\right)\right)}{(m-1)!\Lambda^{\sum_i p_l^i}}\prod_{l=1}^{j-1}dZ_l d\theta^{(l)}$$

Isolating the derivative with respect to $\left|\Psi^{\dagger}\left(\theta_m,Z_m\right)\right|^2$ in the first term of (221) allows to write:

$$\sum_{(p_l)^{1\times j}}\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_{l=1}^{j}p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right) \tag{222}$$

$$=\sum_{p_m,(p_l)^{1\times(j-1)}}\left(\frac{\delta}{\delta\left|\Psi^{\dagger}\left(\theta_m,Z_m\right)\right|^2}\right)^{p_m}\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_{l=1}^{j-1}p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j-1}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)$$

We replace (222) in (221) and change of variable $j-1\rightarrow j$. Then we gather (220) and (221) to obtain:

$$\begin{aligned}\Gamma\left(\Psi^{\dagger},\Psi\right) &= \hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\left(1+\sum_{p_m}\left(\frac{\sharp_{j+1,m}\left(\left(p_l^m\right),\left(p_l^i\right)\right)}{\overline{\sharp}_{j+1,m}\left(\left(p_l^m\right),\left(p_l^i\right)\right)}\left(\frac{\delta}{\delta\left|\Psi^{\dagger}\left(\theta_m,Z_m\right)\right|^2}\right)^{p_m}\right)\right)\\ &\times\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_{l=1}^{j}p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)\\ &\times\int\Psi^{\dagger}\left(\theta_m,Z_m\right)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}\left(\theta_m,Z_m\right),\Psi\left(\theta_m,Z_m\right)\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^m}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta_m,Z_m\right)\\ &\times\int a_{j,m}\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m-1}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\right]\frac{\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)}{(m-1)!\Lambda^{\sum_i p_l^i}}dZ_l d\theta^{(l)}\end{aligned}\tag{223}$$

This relation can be further iterated and we find:

$$\begin{aligned}&\Gamma\left(\Psi^{\dagger},\Psi\right)\\ =\ &\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\int\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\\ &\prod_{i=1}^{m}\left\{\left(\int_{\theta_i<\theta_{i+1}}\Psi^{\dagger}\left(\theta_i,Z_i\right)\frac{\delta^{\sum_l p_l^i}\left[L\left(\Psi^{\dagger}\left(\theta_i,Z_i\right),\Psi\left(\theta_i,Z_i\right)\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta_i,Z_i\right)\right)\right.\\ &\left.\times\left(1+\sum_{p_i}\left(\frac{\sharp_{j+1,i}\left(\left(p_i\right),\left(p_l^i\right)\right)}{\overline{\sharp}_{j+1,i}\left(\left(p_i\right),\left(p_l^i\right)\right)}\left(\frac{\delta}{\delta\left|\Psi^{\dagger}\left(\theta_i,Z_i\right)\right|^2}\right)^{p_i}\right)\right)\right\}\\ &\times a_{j,m}\left(\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_{l=1}^{j}p_l}\left[L\left(\Psi^{\dagger}\left(\theta,Z\right),\Psi\left(\theta,Z\right)\right)\right]}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)\right)\frac{\left(\prod_{l=1}^{j}\Psi\left(\theta^{(l)},Z_l\right)\right)}{(m-1)!\Lambda^{\sum_i p_l^i}}\prod_{i=1}^{m}d\theta_i dZ_i\prod_{l=1}^{j}d\theta^{(l)}dZ_l\end{aligned}\tag{224}$$

3.3.5 Limit of slowly time dependent fields

In the limit of slowly time dependent fields $\Psi(\theta_i, Z_i)$ and of a potential slowly varying in field $\Psi(\theta_i, Z_i)$, we have:

$$\begin{aligned}
&\int_{\theta_i<\theta_{i+1}} \Psi^{\dagger}(\theta_i, Z_i) \frac{\delta^{\sum_l p_l^i}\left[L\left(\Psi^{\dagger}(\theta_i, Z_i), \Psi(\theta_i, Z_i)\right)\right]}{\prod_{l=1}^{j} \delta^{p_l^i}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \Psi(\theta_i, Z_i) \\
=& \int_{\theta_i<\theta_{i+1}} \Psi^{\dagger}(\theta_i, Z_i) \left\{ \frac{1}{2} \frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta \omega^{-1}\left(\theta_i, Z_i, |\Psi|^2\right) + U\left(\theta_i, Z_i, |\Psi|^2\right)\right)}{\prod_{l=1}^{j} \delta^{p_l^i}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \right\} \Psi(\theta_i, Z_i) \\
\simeq& \int_{\theta_i<\theta_{i+1}} \Psi^{\dagger}(\theta_i, Z_i) \left\{ \frac{1}{2} \frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta \omega^{-1}\left(\theta_i, Z_i, |\Psi|^2\right)\right)}{\prod_{l=1}^{j} \delta^{p_l^i}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \right\} \Psi(\theta_i, Z_i)
\end{aligned}$$

If the field varies slowly, this expression reduces to:

$$\int_{\theta_i<\theta_{i+1}} \Psi^{\dagger}(\theta_i, Z_i) \frac{\delta^{\sum_l p_l^i}\left[L\left(\Psi^{\dagger}(\theta_i, Z_i), \Psi(\theta_i, Z_i)\right)\right]}{\prod_{l=1}^{j} \delta^{p_l^i}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \Psi(\theta_i, Z_i) \simeq \frac{1}{2} \frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta \omega^{-1}\left(\theta_{i+1}, Z_{i+1}, |\Psi|^2\right)\right)}{\prod_{l=1}^{j} \delta^{p_l^i}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \left|\Psi(\theta_{i+1}, Z_{i+1})\right|^2$$

and equation (224) can also be rewritten in the approximation of slowly varying background fields:

$$\begin{aligned}
&\Gamma\left(\Psi^{\dagger}, \Psi\right) \qquad (225) \\
=& \hat{S}_{cl}\left(\Psi^{\dagger}, \Psi\right) + \sum_{\substack{j\geqslant 1 \\ m\geqslant 1}} \sum_{\substack{p_l, \left(p_l^i\right)_{m\times j} \\ p_l + \sum_i p_l^i \geqslant 2}} \int\int \left(\prod_{l=1}^{j} \Psi^{\dagger}\left(\theta^{(l)}, Z_l\right)\right) \\
&\times \prod_{i=1}^{m} \left(\left(\frac{1}{2} \frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta \omega^{-1}\left(\theta_{i+1}, Z_{i+1}, |\Psi|^2\right)\right)}{\prod_{l=1}^{j} \delta^{p_l^i}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \left|\Psi(\theta_{i+1}, Z_{i+1})\right|^2 \right.\right. \\
&\left.\left.\times \left(1 + \sum_{p_i} \left(\frac{\sharp_{j+1,i}\left((p_i), \left(p_l^i\right)\right)}{\overline{\sharp}_{j+1,i}\left((p_i), \left(p_l^i\right)\right)}\right) \left(\frac{\delta}{\delta\left|\Psi^{\dagger}(\theta_i, Z_i)\right|^2}\right)^{p_i}\right)\right)\right) a_{j,m} \\
&\times \left(\int \Psi^{\dagger}(\theta, Z) \frac{1}{2} \frac{\delta^{\sum_l p_l}\left(\nabla_\theta \omega^{-1}\left(\theta, Z, |\Psi|^2\right)\right)}{\prod_{l=1}^{j} \delta^{p_l}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \Psi(\theta, Z)\right) \frac{\left(\prod_{l=1}^{j} \Psi\left(\theta^{(l)}, Z_l\right)\right)}{(m-1)!\Lambda^{\sum_i p_l^i}} \prod_{i=1}^{m} d\theta_i dZ_i \prod_{l=1}^{j} d\theta^{(l)} dZ_l
\end{aligned}$$

with the convention that $(\theta_{m+1}, Z_{m+1}) = (\theta, Z)$.

3.3.6 Strong field approximation

For relatively strong fields, the derivatives $\frac{\delta}{\delta\left|\Psi^{\dagger}(\theta_i,Z_i)\right|^2}$ are negligible and (225) reduces to:

$$\begin{aligned}
&\Gamma\left(\Psi^{\dagger},\Psi\right) \qquad (226)\\
=\;& \hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\int\int\prod_{i=1}^{m}\left(\int\Psi^{\dagger}\left(\theta_i,Z_i\right)\left\{\frac{1}{2}\frac{\delta^{\sum_l p_l^i}\left(\nabla_{\theta}\omega^{-1}\left(\theta_i,Z_i,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{\sum_l p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right\}\Psi\left(\theta_i,Z_i\right)\right)\\
&\times\left(a_{j,m}\int\Psi^{\dagger}\left(\theta,Z\right)\left\{\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_{\theta}\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right\}\Psi\left(\theta,Z\right)\right)\\
&\times\left(\prod_{l=1}^{j}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2\right)\prod_{i=1}^{m}d\theta_i dZ_i\prod_{l=1}^{j}d\theta^{(l)}dZ_l
\end{aligned}$$

3.3.7 Weak field approximation

For relatively weak fields, the term $\frac{\sharp_{j+1,i}\left(\left(p_i\right),\left(p_l^i\right)\right)}{\bar{\sharp}_{j+1,i}\left(\left(p_i\right),\left(p_l^i\right)\right)}\left(\frac{\delta}{\delta\left|\Psi^{\dagger}(\theta_i,Z_i)\right|^2}\right)$ in:

$$1+\sum_{p_i}\left(\frac{\sharp_{j+1,i}\left(\left(p_i\right),\left(p_l^i\right)\right)}{\bar{\sharp}_{j+1,i}\left(\left(p_i\right),\left(p_l^i\right)\right)}\left(\frac{\delta}{\delta\left|\Psi^{\dagger}\left(\theta_i,Z_i\right)\right|^2}\right)^{p_i}\right)$$

is dominant and:

$$\begin{aligned}
&\int\Psi^{\dagger}\left(\theta_m,Z_m\right)\frac{\delta^{\sum_l p_l^m}\left[L\left(\Psi^{\dagger}\left(\theta_m,Z_m\right),\Psi\left(\theta_m,Z_m\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^m}\delta\left|\Psi\left(\theta^{\left(k_l^i\right)},Z_l\right)\right|^2}\Psi\left(\theta_m,Z_m\right)\\
&\times\left(1+\sum_{p_m}\left(\frac{\sharp_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}{\bar{\sharp}_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}\left(\frac{\delta}{\delta\left|\Psi^{\dagger}\left(\theta_m,Z_m\right)\right|^2}\right)^{p_m}\right)\right)\\
&\times a_{j,m}\left(\int\Psi^{\dagger}\left(\theta,Z\right)\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_{\theta}\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)\right)\\
\to\;&\int\Psi^{\dagger}\left(\theta_m,Z_m\right)\frac{1}{2}\frac{\delta^{\sum_l p^m}\left(\nabla_{\theta}\omega^{-1}\left(\theta_m,Z_m,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l^m}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\frac{\sharp_{j+1,m}\left(\left(p_m,\left(p_l^m\right)\right)\right)}{\bar{\sharp}_{j+1,m}\left(\left(p_i,\left(p_l^m\right)\right)\right)}\frac{\delta}{\delta\left|\Psi^{\dagger}\left(\theta_m,Z_m\right)\right|^2}\\
&a_{j,m}\left(\int\Psi^{\dagger}\left(\theta,Z\right)\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_{\theta}\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)\right)\\
\simeq\;&\frac{\sharp_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}{\bar{\sharp}_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}a_{j,m}\left(\int\Psi^{\dagger}\left(\theta,Z\right)\frac{1}{2}\frac{\delta^{\sum_l p_l^m}\left(\nabla_{\theta}\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l^m}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\frac{1}{2}\frac{\delta^{\sum_l p_l}\left(\nabla_{\theta}\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)\right)
\end{aligned}$$

Iterating this relation yields:

$$\begin{aligned}
&\Gamma\left(\Psi^{\dagger},\Psi\right) \qquad (227)\\
=\;&\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\left(\prod_{i=1}^{m}\frac{\sharp_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}{4\bar{\sharp}_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}\right)\frac{a_{j,m}}{2}\\
&\times\int\int\Psi^{\dagger}\left(\theta,Z\right)\prod_{i=1}^{m}\left\{\frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right\}\frac{\delta^{\sum_l p_l}\left(\nabla_\theta\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi\left(\theta,Z\right)\\
&\times\prod_{l=1}^{j}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2 d\theta^{(l)}dZ_l
\end{aligned}$$

and this reduces to:

$$\begin{aligned}
&\Gamma\left(\Psi^{\dagger},\Psi\right) \qquad (228)\\
=\;&\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)+\sum_{\substack{j\geqslant 1\\ m\geqslant 1}}\sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}}\left(\prod_{i=1}^{m}\frac{\sharp_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}{4\bar{\sharp}_{j+1,m}\left(\left(p_m\right),\left(p_l^m\right)\right)}\right)\frac{a_{j,m}}{2}\\
&\times\int\int\Psi^{\dagger}\left(\theta,Z\right)\frac{\delta^{\sum_{l,i}p_l^i+p_l}}{\prod_{l=1}^{j}\delta^{\sum_i p_l^i+p_l}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\left(\frac{\left(\nabla_\theta\omega^{-1}\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\prod_{l=1}^{j}\delta^{p_l^i}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^{m+1}\Psi\left(\theta,Z\right)\\
&\times\prod_{l=1}^{j}\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2 d\theta^{(l)}dZ_l
\end{aligned}$$

3.4 First order condition and non trivial vacuum

3.4.1 Constant potential

3.4.1.1 First order condition for classical action We consider the saddle point equation at the lowest order in perturbation. Given our assumptions $\bar{\zeta}_{n+1}>0$ for all $n\geqslant 2$, $\zeta^{(n+1)}>0$, $\zeta^{(2)}<0$, the potential:

$$\alpha\int\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2 dZ_i+\sum\frac{\zeta^{(n)}}{n!}\left(\mathcal{G}_0\left(0,Z\right)+\int\left|\Psi\left(\theta_i^{(i)}-\frac{\left|Z_i-Z_j\right|}{c},Z_j\right)\right|^2 dZ_j\right)^n$$

has a minimum for $\alpha<<1$ and for $\left|\zeta^{(2)}\right|$ large. This minimum is reached for a value X_0 of $\int\left|\Psi\left(\theta^{(i)},Z_i\right)\right|^2 dZ_i$. Up to an irrelevant phase, $\Psi_0\left(\theta^{(i)},Z_i\right)=\Psi_0^{\dagger}\left(\theta,Z\right)=\sqrt{\frac{X_0}{V}}$ where V is the volume of the thread.

Moreover the operator $O=\nabla_\theta\frac{\sigma_\theta^2}{2}\left(\nabla_\theta-\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0\right)\right)$ has positive eigenvalues. Developing $\Psi\left(\theta,Z\right)=\sum a_n\Psi_n\left(\theta,Z\right)$ where $\Psi_n\left(\theta,Z\right)$ are the eigenstates of O, the definition of $\Psi^{\dagger}\left(\theta,Z\right)$ (see [48] and [49]) is given by:

$$\sum\bar{a}_n\Psi_n^{\dagger}\left(\theta,Z\right)$$

where $\Psi_n^{\dagger}\left(\theta,Z\right)$ are the eigenstates of the adjoint operator of O. As a consequence

$$\int-\frac{1}{2}\Psi^{\dagger}\left(\theta,Z\right)\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0\right)\right)\right)\Psi\left(\theta,Z\right)$$

is positive, and null for constant $\Psi^{\dagger}(\theta, Z)$ and $\Psi(\theta, Z)$. As a consequence, for $\left|\zeta^{(n)}\right| > \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0)$ the minimum of $\Gamma(\Psi)$ is reached for $\Psi(\theta, Z) = \Psi_0(\theta, Z) + \delta\Psi(\theta, Z)$ and $\Psi^{\dagger}(\theta, Z) = \Psi_0^{\dagger}(\theta, Z) + \delta\Psi^{\dagger}(\theta, Z)$ where $|\delta\Psi(\theta, Z)| << |\Psi_0(\theta, Z)|$ and $\left|\delta\Psi^{\dagger}(\theta, Z)\right| << \left|\Psi_0^{\dagger}(\theta, Z)\right|$. We assume that the successive derivatives of U decrease quickly and we neglect the terms involving $U^{(n)}(X_0)$ for $n \geqslant 3$.

Expanding the potential around $\Psi_0(\theta, Z)$ and setting $V = 1$, yields at the second order:

$$\begin{aligned}
\Gamma\left(\Psi, \Psi^{\dagger}\right) &= -\frac{1}{2}\int \delta\Psi^{\dagger}(\theta, Z)\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)X_0 \\
&\quad -\frac{1}{2}\int \delta\Psi^{\dagger}(\theta, Z)\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)\delta\Psi(\theta, Z) \\
&\quad +\frac{1}{2}\int \delta\Psi^{\dagger}(\theta, Z)\, U''(X_0)\,\delta\Psi(\theta, Z)
\end{aligned}$$

with $|\Psi|^2 = X_0 + \sqrt{X_0}\left(\delta\left(\Psi^{\dagger} + \delta\Psi\right)\right)$. This leads to the first order condition for $\delta\Psi(\theta_1, Z_1)$:

$$\begin{aligned}
0 &= \frac{1}{2}\delta\Psi^{\dagger}(\theta, Z)\left(-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}(J(\theta), \theta, Z, \mathcal{G}_0 + X_0)\right) + U''(X_0)\right) \\
&\quad -\frac{1}{2}\int \delta\Psi^{\dagger}(\theta_1, Z_1)\sqrt{X_0}\left(\nabla_\theta\frac{\delta\omega^{-1}(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0)}{\delta|\Psi(\theta, Z)|^2}\right)X_0 d\theta_1 dZ_1
\end{aligned}$$

with solution $\delta\Psi^{\dagger}(\theta, Z) = 0$. This implies that the first order condition for $\delta\Psi^{\dagger}(\theta, Z)$ becomes:

$$\begin{aligned}
0 &= -\frac{1}{2}\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)X_0 \\
&\quad -\frac{1}{2}\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)\delta\Psi(\theta, Z) \\
&\quad +\frac{1}{2}U''(X_0)\,\delta\Psi(\theta, Z)
\end{aligned} \tag{229}$$

Equation (229) also rewrites:

$$\left(-\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right) + U''(X_0)\right)(\delta\Psi(\theta, Z) + X_0) = U''(X_0)\,X_0 \tag{230}$$

Equation (230) can be used to write $\delta\Psi(\theta, Z)$ as a function of $\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$:

$$\begin{aligned}
\delta\Psi(\theta, Z) &= \left(\frac{\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)}{U''(X_0) - \left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)}\right)X_0 \\
&= -\frac{\nabla_\theta\left(\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{U''(X_0) - \left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)}X_0
\end{aligned} \tag{231}$$

In first approximation, for $U''(X_0) >> 1$ and $\sigma_\theta^2 << 1$, this yields:

$$\begin{aligned}
\delta\Psi(\theta, Z) &\simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)}{U''(X_0) + \nabla_\theta\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)}X_0 \\
&\simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)}{U''(X_0)}X_0
\end{aligned} \tag{232}$$

Note that for a slowly background field $\Psi_0(\theta, Z)$, equation (232) becomes:

$$\delta\Psi(\theta, Z) \simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)}{U''(X_0)}\Psi_0(\theta, Z) \tag{233}$$

3.4.1.2 Solution of classical action's first order condition To solve equations (232) and (233), the dependency of $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ in $|\Psi|^2$ has to be explicited Note that in first approximation, the solution of (233) is:

$$\begin{aligned}\delta\Psi(\theta,Z) &\simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{U''(X_0)}\Psi_0(\theta,Z)\\ &= \frac{\nabla_\theta\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}{U''(X_0)\,\omega^2\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}\Psi_0(\theta,Z)\end{aligned}$$

and this approximation is sufficient as a first approximation.

However, to find a more precise expression for $\delta\Psi(\theta,Z)$, we use (85) that defines $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ at the classical order:

$$\begin{aligned}&\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right) \qquad (234)\\ &= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\right.\\ &\quad\left.\times\left(\mathcal{G}_0(Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

Using (234), the defining equation (232) for $\delta\Psi(\theta,Z)$ becomes:

$$\begin{aligned}&G^{-1}\left(-\frac{U''(X_0)}{X_0}\int^\theta\delta\Psi(\theta,Z)\right)\\ &= \int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)\left(\mathcal{G}_0(Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\end{aligned}$$

This equation can be rewritten in the local approximation:

$$G^{-1}\left(-\frac{U''(X_0)}{X_0}\int^\theta\delta\Psi(\theta,Z)\right)\simeq J(\theta,Z)+\frac{\left(-\Gamma\nabla_\theta+\Gamma'\nabla_Z^2\right)\left(\omega(J,\theta,Z)\left(\mathcal{G}_0(Z)+|\Psi(\theta,Z)|^2\right)\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)} \qquad (235)$$

where Γ and Γ' are defined by:

$$\begin{aligned}\Gamma &= \int\frac{\kappa}{N}\frac{|Z-Z_1|}{c}T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)dZ_1\\ \Gamma' &= \int\frac{\kappa}{N}|Z-Z_1|^2\,T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)dZ_1\end{aligned}$$

At the lowest order in derivatives, equation (235) becomes:

$$\begin{aligned}G^{-1}\left(-\frac{U''(X_0)}{X_0}\int^\theta\delta\Psi(\theta,Z)\right) &\simeq J(\theta,Z)-\frac{\Gamma\nabla_\theta\omega(J,\theta,Z)\left(\mathcal{G}_0(Z_1)+|\Psi(\theta,Z)|^2\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)} \qquad (236)\\ &= J(\theta,Z)-\Gamma\nabla_\theta|\Psi(\theta,Z)|^2+\Gamma\frac{\delta\Psi(\theta,Z)}{\int^\theta\delta\Psi(\theta,Z)}\left(\mathcal{G}_0(Z)+|\Psi(\theta,Z)|^2\right)\\ &\simeq J(\theta,Z)-\Gamma\sqrt{X_0}\nabla_\theta\delta\Psi(\theta,Z)+\Gamma\frac{\mathcal{G}_0(Z)+X_0+\sqrt{X_0}\delta\Psi(\theta,Z)}{\int^\theta\delta\Psi(\theta,Z)}\delta\Psi(\theta,Z)\\ &\simeq J(\theta,Z)+\Gamma\frac{\mathcal{G}_0(Z_1)+X_0}{\int^\theta\delta\Psi(\theta,Z)}\delta\Psi(\theta,Z)\end{aligned}$$

We set:

$$Y = \ln\left(\int \delta\Psi(\theta, Z)\right)$$

and (236) writes:

$$\begin{aligned} G^{-1}\left(-\frac{U''(X_0)}{X_0}\exp Y\right) &= J(\theta, Z) + \Gamma(\mathcal{G}_0(Z_1) + X_0)\nabla_\theta Y \\ &\simeq \langle J\rangle(Z) + \Gamma(\mathcal{G}_0(Z_1) + X_0)\nabla_\theta Y \end{aligned} \tag{237}$$

where $\langle J\rangle(Z)$ is the current averaged over time. The solution of (237) is:

$$\int \delta\Psi(\theta, Z) = \exp(Y) = \exp\left(H^{-1}\left(\frac{\theta}{\Gamma(\mathcal{G}_0(Z_1) + X_0)} + d\right)\right)$$

with:

$$H(Y) = \int \frac{dY}{G^{-1}\left(-\frac{U''(X_0)}{X_0}\exp Y\right) - \langle J\rangle(Z)}$$

and:

$$\begin{aligned} \delta\Psi(\theta, Z) &= \left(G^{-1}\left(-\frac{U''(X_0)}{X_0}\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_0(Z_1) + \sqrt{X_0}\right)} + d\right)\right)\right) - \langle J\rangle(Z)\right) \\ &\quad \times \exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_0(Z_1) + \sqrt{X_0}\right)} + d\right)\right) \end{aligned} \tag{238}$$

The constant d is chosen so that $\lim_{\theta\to\infty}\delta\Psi(\theta, Z) = 0$. For slowly varyng currents, $\langle J\rangle(Z)$ can replaced by $J(\theta, Z)$ in the formula.

Higher order corrections We use the series expansion (72) to compute higher order corrections to the background field equation.

$$\hat{S}_{cl}\left(\Psi^\dagger, \Psi\right) + A \tag{239}$$

with:

$$A = -\sum_{j\geqslant 1}\int \sum_{\substack{m\geqslant 2, \left(p_l^i\right)_{m\times j} \\ \sum_i p_l^i \geqslant 2}} \prod_{l=1}^{j} d\theta^{(l)} dZ_l \left(\frac{\omega^{-1}\left(J, \theta^{(l)}, Z_l\right)}{\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2}\right)^{\sum_i p_l^i} \left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2 \tag{240}$$

$$\times \frac{\prod_{i=1}^{m} \int c\exp\left(-c(\theta_i - \theta_{i,j}) - \alpha\left(\sum_{l=1, p_l^i\neq 0}^{j}\left(\left(c\left(\theta^{(l-1,i)} - \theta^{(l,i)}\right)\right)^2 - \left|Z_{l-1}^{(i)} - Z_l^{(i)}\right|^2\right)\right)\right)\left|\Psi(\theta_i, Z_i)\right|^2 d\theta_i dZ_i}{(-2)^m D^{\sum_{i,l} p_l^i} m! \prod_k (\sharp_k)! \Lambda_1^j \Lambda^{\sum_{i,l} p_l^i}}$$

The corrective term to $\frac{\delta \hat{S}_{cl}\left(\Psi^\dagger, \Psi\right)}{\delta\Psi}$ is $\frac{\delta A}{\delta\Psi}$ evaluated at $\sqrt{X_0} + \delta\Psi$. At first order in $\delta\Psi$ it is given by:

$$\begin{aligned} \frac{\delta A}{\delta\Psi} = -\sum_{j\geqslant 1}\int \sum_{\substack{m\geqslant 2, \left(p_l^i\right)_{m\times j} \\ \sum_i p_l^i \geqslant 2}} &\left[\frac{\prod_{i=1}^{m}\int\left[c\exp\left(-cl_j - \alpha\left(\sum_{r=1}^{n-1}\left(\left(c\left(l_r - l_{r+1}\right)\right)^2 - \left|Z_r - Z_{r+1}\right|^2\right)\right)\right)X_0\right]}{X_0^2 2^m D^{\sum_{i,l} p_l^i} m! \prod_k (\sharp_k)! \Lambda_1^j \Lambda^{\sum_{i,l} p_l^i}}\right. \\ &\left.\times\left(m + \sum_i p_l^i\right)\left(m + \sum_i p_l^i - 1\right) X_0^{j-2} \prod_{l=1}^{j}\left(\frac{\omega_0^{-1}(J, Z_i)}{X_0}\right)^{p_l^i} dl_i dZ_i\right] \times \prod_{l=1}^{j} dl_l dZ_l \times \sqrt{X_0}\delta\Psi \end{aligned}$$

where we have replaced $\omega^{-1}(J, \theta_i - l_l, Z_i)$ by it's static solution approximation $\omega_0^{-1}(J, Z_i)$.

The integral may be computed and yield a constant $-\frac{1}{2}C(X_0)$ times $\delta\Psi$. Equation (229) is thus replaced by:

$$\begin{aligned}0 = &-\frac{1}{2}\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)X_0 \\ &-\frac{1}{2}\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)\delta\Psi(\theta,Z) \\ &+\frac{1}{2}\left(U''(X_0)-C(X_0)\right)\delta\Psi(\theta,Z)\end{aligned} \tag{241}$$

As a consequence, equation (231) is thus transformed into:

$$\begin{aligned}\delta\Psi(\theta,Z) &\simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{U''(X_0)-C(X_0)+\nabla_\theta\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}X_0 \\ &\simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{U''(X_0)-C(X_0)}X_0\end{aligned} \tag{242}$$

Solution (238) is thus still valid, with $U''(X_0)$ shifted: $U''(X_0)\rightarrow U''(X_0)-C(X_0)$.

11.0.1 Time dependent potential

When the system is interacting with external signals, the average number of activations may be shiftted and the assumption of a constant minimum $\sqrt{\frac{X_0}{V}}$ of the potential has to be modified. We consider a time dependent potential with minimum $\Psi_0(\theta,Z)$. Expanding the the effective action at the second order around $\Psi_0(\theta,Z)$ yields:

$$\begin{aligned}\Gamma(\Psi) = &-\frac{1}{2}\int\delta\Psi^\dagger(\theta,Z)\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)\Psi_0(\theta,Z) \\ &-\frac{1}{2}\int\Psi_0^\dagger(\theta,Z)\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)\delta\Psi(\theta,Z) \\ &-\frac{1}{2}\int\delta\Psi^\dagger(\theta,Z)\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)\delta\Psi(\theta,Z) \\ &+\frac{1}{2}\int\delta\Psi^\dagger(\theta,Z)U''(X_0)\delta\Psi(\theta,Z)\end{aligned}$$

The second term can still be neglected for relatively slow variations of $\Psi_0^\dagger(\theta,Z)$. The solutions are thus similar to the previous paragraph:

$$\delta\Psi^\dagger(\theta,Z)=0$$

and:

$$\begin{aligned}\delta\Psi(\theta,Z) &= \left(\frac{\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)}{U''(X_0)-\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)}\right)\Psi_0(\theta,Z) \\ &\simeq -\frac{\nabla_\theta\left(\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)}{U''(X_0)-\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)}\Psi_0(\theta,Z) \\ &\simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{U''(X_0)+\nabla_\theta\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}\Psi_0(\theta,Z)\end{aligned}$$

Appendix 4. Correlation functions and corrections to frequencies equations

4.1 Two points correlation functions

4.1.1 Second order effective vertex

The two points Green function is the inverse of the second derivative of the effective action $\Gamma\left(\Psi^{\dagger},\Psi\right)$, defined by:

$$\Gamma_{1,1}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right)\right)=\frac{\delta^{2}\Gamma\left(\Psi^{\dagger},\Psi\right)}{\delta\Psi^{\dagger}\left(\theta_{f},Z_{f}\right)\delta\Psi\left(\theta_{i},Z_{i}\right)}$$

where, in first approximation:

$$\begin{aligned}&\Gamma_{1,1}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right)\right)\\=&-\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)+\hat{\Gamma}_{1,1}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right)\right)\end{aligned}$$

with $\hat{\Gamma}_{1,1}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right)\right)$ given by the second derivatives of (57). We decompose this vertex in two parts:

$$\hat{\Gamma}_{1,1}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right)\right)=\hat{\Gamma}_{1,1}^{(1)}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right)\right)+\hat{\Gamma}_{1,1}^{(2)}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right)\right)$$

where:

$$\begin{aligned}&\hat{\Gamma}_{1,1}^{(1)}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right),\Psi^{\dagger},\Psi\right) \quad (243)\\=&\sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_{l}^{i}\right)_{m\times j}\\ \sum_{i}p_{l}^{i}\geqslant 2}}\sum_{l_{f}^{\prime}=1,l_{i}^{\prime}=1}^{j}\int\frac{\left(\prod_{l=1,l\neq l_{f}^{\prime}}^{j}\Psi^{\dagger}\left(\theta_{f}^{(l)},Z_{l}\right)\right)}{m!\prod_{k}\left(\sharp_{k}\right)!}\times\frac{\prod_{l=1}^{j}\exp\left(-\Lambda_{1}\left(\theta_{f}^{(l)}-\theta_{i}^{(l)}\right)\right)}{\Lambda^{\sum_{i,l}p_{l}^{i}}}\\&\times\prod_{i=1}^{m}\left[\int_{\prod_{l=1}^{j}\left[\theta_{i}^{(l)},\theta_{f}^{(l)}\right]^{p_{l}^{i}}}\frac{\delta^{\sum_{l}p_{l}^{i}}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{i}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{i}}d\theta^{\left(k_{l}^{i}\right)}\right]_{\substack{\left(\theta^{\left(l_{f}^{\prime}\right)},Z_{l_{f}^{\prime}}\right)=\left(\theta_{f},Z_{f}\right)\\ \left(\theta^{\left(l_{i}^{\prime}\right)},Z_{l_{i}^{\prime}}\right)=\left(\theta_{i},Z_{i}\right)}}\times\left(\prod_{l=1,l\neq l_{i}^{\prime}}^{j}\Psi\left(\theta_{i}^{(l)},Z_{l}\right)\right)\end{aligned}$$

and:

$$\begin{aligned}&\hat{\Gamma}_{1,1}^{(1)}\left(\left(\theta_{f},Z_{f}\right),\left(\theta_{i},Z_{i}\right),\Psi^{\dagger},\Psi\right) \quad (244)\\=&\sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_{l}^{i}\right)_{m\times j}\\ \sum_{i}p_{l}^{i}\geqslant 2}}\int\Psi^{\dagger}\left(\theta_{f},Z_{f}\right)\left(\prod_{l=1}^{j}\Psi^{\dagger}\left(\theta_{f}^{(l)},Z_{l}\right)\right)\\&\times\frac{\delta^{2}}{\delta\left|\Psi\left(\theta_{f},Z_{f}\right)\right|^{2}\delta\left|\Psi\left(\theta_{i},Z_{i}\right)\right|^{2}}\prod_{i=1}^{m}\left[\int_{\prod_{l}\left[\theta_{i}^{(l)},\theta_{f}^{(l)}\right]^{p_{l}^{i}}}\frac{\delta^{\sum_{l}p_{l}^{i}}\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{i}}\delta\left|\Psi\left(\theta^{\left(k_{l}^{i}\right)},Z_{l}\right)\right|^{2}}\prod_{l=1}^{j}\prod_{k_{l}^{i}=1}^{p_{l}^{i}}d\theta^{\left(k_{l}^{i}\right)}\right]\\&\times\frac{\prod_{l=1}^{j}\exp\left(-\Lambda_{1}\left(\theta_{f}^{(l)}-\theta_{i}^{(l)}\right)\right)}{m!\prod_{k}\left(\sharp_{k}\right)!\Lambda^{\sum_{i,l}p_{l}^{i}}}\left(\prod_{l=1}^{j}\Psi\left(\theta_{i}^{(l)},Z_{l}\right)\right)\Psi\left(\theta_{i},Z_{i}\right)\end{aligned}$$

The second contribution corresponds to the influence of the propagation over the whole system and can be neglected, as said in the text.

To find an approximate series expansion of $\hat{\Gamma}_{1,1}((\theta_f,Z_f),(\theta_i,Z_i))$, we work with the local series expansion (151) of $\hat{\Gamma}_{1,1}\left(\Psi^{\dagger},\Psi\right)$. Using (72) alongside with the associated notations, we can rewrite (151) as:

$$\begin{aligned}
&\hat{\Gamma}_{1,1}\left((\theta,Z_f),(\theta,Z_i),\Psi^{\dagger},\Psi\right) \qquad (245)\\
=&\sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\sum_{\substack{\left(p_l^i\right)_{m\times j}\\ \sum_i p_l^i\geqslant 2}}\sum_{l'_f=1,l'_i=1}^{j}\int\Bigg[\left(\prod_{l=1,l\neq l'_f}^{j}\Psi^{\dagger}\left(\theta^{(l)},Z_l\right)\right)\times\\
&\frac{\prod_{i=1}^{m}\int c\exp\left(-c\left(\theta_i-\theta_{i,j}\right)-\alpha\left(\sum_{l=1,p_l^i\neq 0}^{j}\left(\left(c\left(\theta^{(l-1,i)}-\theta^{(l,i)}\right)\right)^2-\left|Z_{l-1}^{(i)}-Z_l^{(i)}\right|^2\right)\right)\right)\left|\Psi\left(\theta_i,Z_i\right)\right|^2 d\theta_i dZ_i}{(-2)^m D^{\sum_{i,l}p_l^i}m!\prod_k(\sharp_k)!\Lambda_1^j\Lambda^{\sum_{i,l}p_l^i}}\\
&\left(\frac{\omega^{-1}\left(J,\theta^{(l)},Z_l\right)}{\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right)^{\sum_i p_l^i}\left(\prod_{l=1,l\neq l'_i}^{j}\Psi\left(\theta^{(l)},Z_l\right)d\theta^{(l)}dZ_l\right)\Bigg]_{\substack{\left(\theta^{(l')},Z_{l'_f}\right)=(\theta_i,Z_f)\\ \left(\theta^{(l')},Z_{l'_i}\right)=(\theta_f,Z_i)}}
\end{aligned}$$

For $\Lambda>>1$, the dominant contributions is obtained for each point connected by two vertices of valence 2. As a consequence $\prod_k(\sharp_k)!=j$. Moreover there are $j(j-1)$ possibilities to choose the external points (dismissing the same point) and $m(m-1)$ possibilities to differentiate with respect to $\left|\Psi\left(\theta_i,Z_i\right)\right|^2$ and $\left|\Psi\left(\theta_f,Z_f\right)\right|^2$. For internal points that are integrated on, we can replace the sums over $\left(\theta^{(l)},Z_l\right)$ by the values of the quantities evaluated at their average $\left(\bar{\theta},\bar{Z}\right)$ and we have:

$$\begin{aligned}
&\hat{\Gamma}_{1,1}\left((\theta,Z_f),(\theta,Z_i),\Psi^{\dagger},\Psi\right) \qquad (246)\\
\simeq&\sum_{\substack{j\geqslant 2\\ m\geqslant 2}}\int\Bigg[d\bar{\theta}d\bar{Z}\left(\left|\Psi\left(\bar{\theta},\bar{Z}\right)\right|^2\right)^{j-2}\left(\frac{\omega^{-1}\left(\bar{\theta},\bar{Z}\right)}{\left|\Psi\left(\bar{\theta},\bar{Z}\right)\right|^2}\right)^{j-2}\Psi^{\dagger}\left(\theta_i,Z_i\right)\Psi\left(\theta_f,Z_f\right)\frac{\omega^{-1}\left(\theta_i,Z_i\right)}{\left|\Psi\left(\theta_i,Z_i\right)\right|^2}\frac{\omega^{-1}\left(\theta_f,Z_f\right)}{\left|\Psi\left(\theta_f,Z_f\right)\right|^2}\\
&\times\frac{\left(\int c\exp\left(-c\left(\theta-\bar{\theta}\right)-\alpha\left(\left(c\left(\theta-\bar{\theta}\right)\right)^2-\left|Z-\bar{Z}\right|^2\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2 d\theta dZ\right)^{m-2}}{(-2)^m D^m(m-2)!(j-2)!\Lambda_1^j\Lambda^{2j}}\\
&\times\frac{\left(\int c\exp\left(-c\left(\theta-\theta_i\right)-\alpha\left(\left(c\left(\theta-\theta_i\right)\right)^2-\left|Z-Z_i\right|^2\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2 d\theta dZ\right)}{(-2)D} \qquad (247)\\
&\times\frac{\left(\int c\exp\left(-c\left(\theta-\theta_f\right)-\alpha\left(\left(c\left(\theta-\theta_f\right)\right)^2-\left|Z-Z_f\right|^2\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2 d\theta dZ\right)}{(-2)D} \qquad (248)
\end{aligned}$$

We can replace the exponentials:

$$\int c\exp\left(-c\left(\theta-\bar{\theta}\right)-\alpha\left(\left(c\left(\theta-\bar{\theta}\right)\right)^2-\left|Z-\bar{Z}\right|^2\right)\right)\left|\Psi\left(\theta_i,Z_i\right)\right|^2 d\theta_i dZ_i$$

by its dominant contribution:

$$\int c\exp\left(-c\left(\theta-\bar{\theta}\right)\right)\left|\Psi\left(\theta_i,\bar{Z}+c\left(\theta-\bar{\theta}\right)\mathbf{e}\right)\right|^2 d\theta d\mathbf{e}$$

where $\mathbf{e}$ is a unit vector. As a consequence:

$$\begin{aligned}
&\hat{\Gamma}_{1,1}\left(\left(\theta,Z_f\right),\left(\theta,Z_i\right),\Psi^{\dagger},\Psi\right)\\
\simeq\;&\frac{\omega^{-1}\left(\theta_i,Z_i\right)}{\Psi\left(\theta_i,Z_i\right)}\frac{\omega^{-1}\left(\theta_f,Z_f\right)}{\Psi^{\dagger}\left(\theta_f,Z_f\right)}\exp\left(\int\frac{\omega^{-1}\left(\bar{\theta},\bar{Z}\right)}{\Lambda_1}d\bar{\theta}d\bar{Z}\right)\\
&\times\exp\left(-\int\frac{c}{2D\Lambda}\exp\left(-c\left(\theta-\bar{\theta}\right)\right)\left|\Psi\left(\theta,\bar{Z}+c\left(\theta-\bar{\theta}\right)\mathbf{e}\right)\right|^2 d\theta d\mathbf{e}\right)\\
&\times\left(\int\frac{c}{2D\Lambda}\exp\left(-c\left(\theta-\theta_i\right)\right)\left|\Psi\left(\theta,Z_i+c\left(\theta-\theta_i\right)\mathbf{e}\right)\right|^2 d\theta d\mathbf{e}\right)\\
&\times\left(\int\frac{c}{2D\Lambda}\exp\left(-c\left(\theta-\theta_f\right)\right)\left|\Psi\left(\theta,Z_f+c\left(\theta-\theta_f\right)\mathbf{e}\right)\right|^2 d\theta d\mathbf{e}\right)\\
\simeq\;&\frac{\omega^{-1}\left(\theta_i,Z_i\right)}{\Psi\left(\theta_i,Z_i\right)}\frac{\omega^{-1}\left(\theta_f,Z_f\right)}{\Psi^{\dagger}\left(\theta_f,Z_f\right)}\exp\left(\int\frac{\omega^{-1}\left(\bar{\theta},\bar{Z}\right)}{\Lambda_1}d\bar{\theta}d\bar{Z}\right)\times\exp\left(-\int\frac{1}{2D\Lambda}\left|\Psi\left(\bar{\theta},\bar{Z}\right)\right|^2 d\bar{\theta}d\bar{Z}\right)\\
&\times\frac{1}{2D\Lambda}\left|\Psi\left(\theta_i,Z_i\right)\right|^2\times\frac{1}{2D\Lambda}\left|\Psi\left(\theta_f,Z_f\right)\right|^2\\
=\;&\omega^{-1}\left(\theta_i,Z_i\right)\omega^{-1}\left(\theta_f,Z_f\right)\Psi\left(\theta_f,Z_f\right)\Psi^{\dagger}\left(\theta_i,Z_i\right)C\left(\bar{\omega},\Psi\right)
\end{aligned}$$

We thus have:

$$\hat{\Gamma}_{1,1}\left(\left(\theta,Z_f\right),\left(\theta,Z_i\right),\Psi^{\dagger},\Psi\right)=\omega^{-1}\left(\theta_i,Z_i\right)\omega^{-1}\left(\theta_f,Z_f\right)\Psi\left(\theta_f,Z_f\right)\Psi^{\dagger}\left(\theta_i,Z_i\right)C\left(\bar{\omega},\Psi\right)$$

where:

$$C\left(\bar{\omega},\Psi\right)=\frac{1}{2D\Lambda}\frac{1}{2D\Lambda}\exp\left(\int\frac{\omega^{-1}\left(\bar{\theta},\bar{Z}\right)}{\Lambda_1}d\bar{\theta}d\bar{Z}\right)\times\exp\left(-\int\frac{c}{2D\Lambda}\left|\Psi\left(\bar{\theta},\bar{Z}\right)\right|^2 d\bar{\theta}d\bar{Z}\right)$$

4.1.2 2 points correlation function

Inverting $\Gamma_{1,1}\left(\theta_f,\theta_i\right)$ yields the two-points Green function:

$$G_2\left(\theta_f,\theta_i\right)=\mathcal{G}\left(\theta_f,\theta_i\right)+\mathcal{G}*\sum_{n\geqslant 2}(-1)^{n-1}\left(\hat{\Gamma}_{1,1}\left(\left(\theta_f,Z_f\right),\left(\theta_i,Z_i\right),\Psi^{\dagger},\Psi\right)*\mathcal{G}\right)^n$$

where:

$$-\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta}-\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\right)\mathcal{G}\left(\theta_f,\theta_i,Z_f,Z_i\right)=\delta\left(\theta_f-\theta_i\right)\delta\left(Z_f-Z_i\right)\tag{249}$$

For relatively slow variations in frequencies, we can use (167) by replacing $\omega^{-1}\left(J,\theta,Z,\mathcal{G}_0\right)$ by its average on the interval $[\theta,\theta']$:

$$\omega^{-1}\left(J,\theta,Z,\mathcal{G}_0\right)=\frac{1}{\bar{X}_r}\to\left\langle\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)\right\rangle_{[\theta,\theta']}\equiv\bar{\omega}^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_0+\left|\Psi\right|^2\right)$$

in (164):

$$\mathcal{G}\left(\theta_f,\theta_i,Z_f,Z_i\right)=\delta\left(Z_f-Z_i\right)\frac{\exp\left(-\left(\sqrt{\left(\frac{\bar{\omega}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\sigma^2}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{\bar{\omega}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\sigma^2}\right)\left(\theta-\theta'\right)\right)}{\sqrt{\frac{\pi}{2}}\sqrt{\left(\frac{\bar{\omega}^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\sigma^2}\right)^2+\frac{2\alpha}{\sigma^2}}}H\left(\theta-\theta'\right)$$

As a consequence, the solution of (249) is a series expansion:

$$
\begin{aligned}
G_2\left(\theta_f, \theta_i, Z_f, Z_i\right) &= \mathcal{G}\left(\theta_f, \theta_i, Z_f, Z_i\right) + \mathcal{G} * \sum_{n \geqslant 2} (-1)^{n-1} \left(\hat{\Gamma}_{1,1} * \mathcal{G}\right)^n \\
&= \mathcal{G}\left(\theta_f, \theta_i, Z_f, Z_i\right) + \int \left(\prod_{k=1}^{n} d\theta_k dZ_k\right) \mathcal{G}\left(\theta_f, \theta_n, Z_f, Z_f\right) \frac{\Psi\left(\theta_n, Z_f\right)}{\omega\left(\theta_n, Z_f\right)} \times \sum_{n \geqslant 2} (-1)^{n-1} \\
&\quad \times \prod_{k=1}^{n-1} \left(\left|\Psi\left(\theta_k, Z_k\right)\right|^2 \left(\omega^{-1}\left(\theta_k, Z_k\right)\right)^2 \mathcal{G}\left(\theta_{k+1}, \theta_k, Z_k, Z_k\right)\right) \frac{\Psi^{\dagger}\left(\theta_1, Z_i\right)}{\omega\left(\theta_1, Z_i\right)} \mathcal{G}\left(\theta_1, \theta_i, Z_i, Z_i\right)
\end{aligned}
$$

4.2 (k,n) points correlation functions

The (k,n) points correlation functions, are found using the standard techniques. We first derive the (k,n) effective vertex, the associated connected correlation function, from which the (k,n) correlation function is derived.

4.2.1 (k,n) effective vertices

The (k,n)-th effective vertex are defined by:

$$
\Gamma_{k,n}\left(\left(\theta_f^{(l)}, Z_l\right)_{l=1,..,k}, \left(\theta_i^{(l)}, Z_l\right)_{l=1,..,n}\right) = \frac{\delta^{k+n} \Gamma\left(\Psi^{\dagger}, \Psi\right)}{\delta^k \left(\Psi^{\dagger}\left(\theta_f^{(l)}, Z_l\right)\right)_{l=1,..,k} \delta^n \left(\Psi\left(\theta_i^{(l)}, Z_l\right)\right)_{l=1,..,n}}
$$

Neglecting the derivatives corresponding to the impact of propagation between θ_i and θ_f, we find:

$$
\begin{aligned}
&\Gamma_{k,n}\left(\left(\theta_f^{(r)}, Z_r\right)_{r=1,..,k}, \left(\theta_i^{(s)}, Z_s\right)_{s=1,..,n}\right) \\
= &\sum_{\substack{j \geqslant \max(k,n) \\ m \geqslant 2}} \sum_{\substack{\left(p_l^i\right)_{m \times j} \\ \sum_i p_l^i \geqslant 2}} \sum_{\substack{\left(l_1' \ldots, l_k'\right) \\ \times \left(l_1'' \ldots, l_n''\right) \\ \subset (1, \ldots, j)^2}}^{j} \int \left(\prod_{l=1, l \notin \left(l_i'\right)}^{j} \Psi^{\dagger}\left(\theta_f^{(l)}, Z_l\right)\right) \\
&\times \prod_{i=1}^{m} \left[\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)}, \theta_f^{(l)}\right]^{p_l^i}} \frac{\delta^{\sum_l p_l^i} \left[\hat{S}_{cl}\left(\Psi^{\dagger}, \Psi\right)\right]}{\prod_{l=1}^{j} \prod_{k_l^i=1}^{p_l^i} \delta \left|\Psi\left(\theta^{\left(k_l^i\right)}, Z_l\right)\right|^2} \prod_{l=1}^{j} \prod_{k_l^i=1}^{p_l^i} d\theta^{\left(k_l^i\right)}\right]_{\substack{\left(\theta^{\left(l_i'\right)}, Z_{l_i'}\right) = \left(\theta_f^{(r)}, Z_r\right) \\ \left(\theta^{\left(l_i''\right)}, Z_{l_i''}\right) = \left(\theta_i^{(s)}, Z_s\right)}} \\
&\times \frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)} - \theta_i^{(l)}\right)\right)}{m! \prod_k \left(\sharp_k\right)! \Lambda^{\sum_i p_l^i}} \left(\prod_{l=1, l \notin \left(l_i''\right)}^{j} \Psi\left(\theta_i^{(l)}, Z_l\right)\right)
\end{aligned}
$$

Under the approximation that $\frac{\delta^p\left[\hat{S}_{cl}\left(\Psi^\dagger,\Psi\right)\right]}{\prod_{i=1}^{p}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}$ is decreasing with p, the previous expression reduces to:

$$\Gamma_{k,n}\left(\left(\theta_f^{(r)},Z_r\right)_{r=1,..,k},\left(\theta_i^{(s)},Z_s\right)_{s=1,..,n}\right) \tag{250}$$
$$= \sum_{\substack{j=\max(k,n)\\ m\geqslant 2}}\ \sum_{\substack{\cup L_i\\ =2\{1,...,j\}}}\ \sum_{\substack{(l_1',...,l_k')\\ \times(l_1'',...,l_n'')\\ \subset(1,...,j)^2}}^{j}\int\frac{\left(\prod_{l=1,l\notin(l_i')}^{j}\Psi^\dagger\left(\theta_f^{(l)},Z_l\right)\right)}{\Lambda^{2j}}$$
$$\times\int_{\prod_{l=1}^{j}\left[\theta_i^{(l)},\theta_f^{(l)}\right]^2}\left[\prod_{i=1}^{m}\frac{\delta^{\sharp L_i}\left[\hat{S}_{cl}\left(\Psi^\dagger,\Psi\right)\right]}{\prod_{l_i\in L_i}\delta\left|\Psi\left(\theta^{(l_i)},Z_{l_i}\right)\right|^2}\prod_{l_i\in L_i}d\theta^{(l_i)}\right]_{\substack{\left(\theta^{(l_i')},Z_{l_i'}\right)=\left(\theta_f^{(r)},Z_r\right)\\ \left(\theta^{(l_i'')},Z_{l_i''}\right)=\left(\theta_i^{(s)},Z_s\right)}}$$
$$\times\frac{\exp\left(-\Lambda_1\left(\theta_f^{(l)}-\theta_i^{(l)}\right)\right)}{m!\prod_k(\sharp_k)!}\left(\prod_{l=1,l\notin(l_i'')}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)$$

In first approximation:

$$\frac{\delta^{\sharp L_i}\left[\hat{S}_{cl}\left(\Psi^\dagger,\Psi\right)\right]}{\prod_{l_i\in L_i}\delta\left|\Psi\left(\theta^{(l_i)},Z_{l_i}\right)\right|^2}\simeq\prod_{l_i\in L_i}\nabla_{\theta_i}\omega^{-1}\left(J\left(\theta^{(l_i)}\right),\theta^{(l_i)},Z_{l_i},\mathcal{G}_0+|\Psi|^2\right)$$

where $2\{1,...,j\}$ denotes two copies of $\{1,...,j\}$ and (**??**) gives:

$$\nabla_{\theta_i}\omega^{-1}\left(\theta^{(i)},Z\right) \tag{251}$$
$$= \nabla_{\theta_i}\frac{G'[J,\omega,\theta,Z,\Psi]}{F'[J,\omega,\theta,Z,\Psi]}\int\sum_{k=0}^{\infty}\frac{\exp\left(-\alpha\left(\theta_2-\theta_1-\sum_{l=0}^{k}\frac{|Z_l-Z_{l+1}|}{c}\right)\right)}{A^{k+1}}$$
$$\times\left(\prod_{l=1}^{k}\int\left(|\Psi(\theta-l_l,Z_l)|^2-\frac{\left(\hat{T}_1\omega_0|\Psi|^2\right)(\theta-l_l,Z_l)}{\left(\omega_0+\hat{T}\omega_0|\Psi|^2\right)(\theta-l_l,Z_l)}\right)dZ_l dl_l\right)\omega_0(J,\theta-l_k,Z_k)|\Psi(\theta-l_k,Z_k)|^2$$

4.2.2 (k,n) connected correlation functions

The link between connected correlation functions and effective vertices is obtained by the recursive relation:

$$G_{k,n}^{(c)}\left(\left(\theta_f^{(r)},Z_r\right)_{r=1,..,k},\left(\theta_i^{(s)},Z_s\right)_{s=1,..,n}\right) = -\Gamma_{k,n}\left(\left(\theta_f^{(r)},Z_r\right)_{r=1,..,k},\left(\theta_i^{(s)},Z_s\right)_{s=1,..,n}\right)$$
$$+\sum_l(-1)^l\sum_{\substack{\sum k_i=k+l\\ \sum n_i=n+l}}\sum_G\left[\Gamma_{k_1,n_1}*_{\mathcal{G}}...*_{\mathcal{G}}\Gamma_{k_l,n_l}\right]_G$$

where the sum over G denotes the graphs with trivial fundamental group obtained by drawing l vertices labelled Γ_{k_i,n_i}. The vertex Γ_{k_i,n_i} has two types of valences, denoted in and out, of order (k_1,n_1). The Γ_{k_i,n_i} are connected by segments issued from a valence of type out to a valence of type in. Only (k,l) valences labelled $\left(\theta_f^{(r)},Z_r\right)_{r=1,..,k},\left(\theta_i^{(s)},Z_s\right)_{s=1,..,n}$ are left free.

The expression $\left[\Gamma_{k_1,n_1} *_{\mathcal{G}} ...*_{\mathcal{G}}\Gamma_{k_l,n_l}\right]_G$ is computed for each graph G by associating a propagator $\mathcal{G}$ to each leg of the graph: and convoluting all expressions that are connected. For slowly varying $\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)$ in fields, $\left|\frac{\delta^p\left[\hat{S}_{cl}\left(\Psi^{\dagger},\Psi\right)\right]}{\prod\limits_{i=1}^{p}\delta\left|\Psi\left(\theta^{(k_i)},Z_{k_i}\right)\right|^2}\right| << 1$ and in first approximation:

$$G_{k,n}^{c}\left(\left(\theta_f^{(r)},Z_r\right)_{r=1,..,k},\left(\theta_i^{(s)},Z_s\right)_{s=1,..,n}\right)\simeq -\Gamma_{k,n}\left(\left(\theta_f^{(r)},Z_r\right)_{r=1,..,k},\left(\theta_i^{(s)},Z_s\right)_{s=1,..,n}\right)$$

4.2.3 (k,n) correlation functions

The Green functions can ultimately be computed in the usual way. They are defined by:

$$\begin{aligned}&G_{k,n}\left(\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k},\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}\right)\\ =&\sum_{i=1,j=1}^{k,n}\sum_{P_i(k),P_j(n)}\prod_{\substack{r\in P_i(k)\\ s\in P_j(n)}}G_{k_r,n_s}^{(c)}\left(\left(\theta_f^{(l,r)},Z_{l,r}\right)_{l=1,..,k_r},\left(\theta_i^{(l,s)},Z_{l,s}\right)_{l=1,..,n_s}\right)\end{aligned}$$

where $P_i(k)$ and $P_j(n)$ denote the partitions of k and n in i and j subsets and:

$$\begin{aligned}\cup_r\left(\theta_f^{(l,r)},Z_{l,r}\right)_{l=1,..,k_r}&=\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k}\\ \cup_s\left(\theta_i^{(l,s)},Z_{l,s}\right)_{l=1,..,n_s}&=\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}\end{aligned}$$

In first approximation, the Green function can thus be written:

$$\begin{aligned}&G_{k,n}\left(\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k},\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}\right)\\ =&\sum_{\sigma_k,\sigma_n}\sum_{u=0}^{\inf(k,n)}\prod_{j=0}^{u}G_2\left(\left(\theta_f^{(j)},Z_f\right),\left(\theta_i^{(i)},Z_i\right)\right)\\ &\times\sum_{i=1,j=1}^{k-u,n-u}(-1)^{i+j}\sum_{\substack{P_i(k-u)\\ P_j(n-u)}}\prod_{\substack{r\in P_i(k-u)\\ s\in P_j(n-u)}}\Gamma_{k_r,n_s}\left(\left(\theta_f^{(l,r,u)},Z_{l,r,u}\right)_{l=1,..,k_r},\left(\theta_i^{(l,s,u)},Z_{l,s,u}\right)_{l=1,..,n_s}\right)\end{aligned}$$

where $\cup_r\left(\theta_f^{(l,r,u)},Z_{l,r,u}\right)_{l=1,..,k_r}=\left(\theta_f^{(l)},Z_l\right)_{l=u+1,..,k}$ and $\cup_s\left(\theta_i^{(l,s,u)},Z_{l,s,u}\right)_{l=u+1,..,n_s}=\left(\theta_i^{(l)},Z_l\right)_{l=u+1,..,n}$ as ordered sets and the sum over σ_k and σ_n is over all permutations of the $\left(\theta_f^{(l)},Z_l\right)_{l=1,..,k}$ and $\left(\theta_i^{(l)},Z_l\right)_{l=1,..,n}$ respectively.

4.3 Corrections to (97)

4.3.1 Series expansion

The corrective terms to $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ are obtained by isolating $\Omega(\theta,Z)$ in (82):

$$\Gamma\left(\Psi^{\dagger},\Psi\right)\simeq\int\Psi^{\dagger}(\theta,Z)\left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta}-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\delta\left(\theta_f-\theta_i\right)+\Omega(\theta,Z)\right)\Psi(\theta,Z)$$

and by integration over θ (see (84)). As explained in the text, we study the weak field case in the local approximation, so that we use formula (228) for the effective action. Combining (228) with (84) we obtain $\omega_e^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$:

$$\omega_e^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)=\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)+Z\tag{252}$$

where:

$$Z = \int^{\theta} d\theta \sum_{\substack{j\geqslant 1\\ m\geqslant 1}} \sum_{\substack{p_l,\left(p_l^i\right)_{m\times j}\\ p_l+\sum_i p_l^i\geqslant 2}} \left(\prod_{i=1}^{m} \frac{\sharp_{j+1,m}\left(\left(p_m,\left(p_l^m\right)\right)\right)}{4\bar{\sharp}_{j+1,m}\left(\left(p_i,\left(p_l^m\right)\right)\right)}\right) \frac{a_{j,m}}{2} \tag{253}$$
$$\times \int \prod_{i=1}^{m} \left\{ \frac{\delta^{\sum_l p_l^i}\left(\nabla_\theta \omega^{-1}\left(\theta, Z, |\Psi|^2\right)\right)}{\prod_{l=1}^{j} \delta^{p_l^i}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \right\} \frac{\delta^{\sum_l p_l}\left(\nabla_\theta \omega^{-1}\left(\theta, Z, |\Psi|^2\right)\right)}{\prod_{l=1}^{j} \delta^{p_l}\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \prod_{l=1}^{j} \left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2 d\theta^{(l)} dZ_l$$

Equation (252) is a series in $\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$. We provide below a detailed computation of the lowest order terms.

To conclude, note that the series can be divided in two different parts and writes:

$$Z = \left[\frac{1}{n!}\left(\sum_{l\geqslant 2} \frac{1}{l!} \int \Psi^\dagger(\theta_i, Z_i)\Psi(\theta_i, Z_i) \frac{\delta^l}{\Lambda^l \delta^l \left|\Psi(\theta_i, Z_i)\right|^2} d\theta_i\right)^n \right.$$
$$\left. \times \sum \frac{1}{p!}\left(\nabla_\theta \omega^{-1}\left(\theta, Z, |\Psi|^2\right) + \Psi^\dagger(\theta, Z)\nabla_\theta \omega^{-1}\left(\theta, Z, |\Psi|^2\right)\Psi(\theta, Z)\right)^p\right]_{\Psi(\theta,Z)=\Psi^\dagger(\theta,Z)=0}$$

and that this can also be written by using the exponential form given in (203) in the local approximation:

$$Z = \left[\exp\left(\sum_{l\geqslant 2} \frac{1}{l!} \int \Psi^\dagger(\theta_i, Z_i)\Psi(\theta_i, Z_i) \frac{\delta^l}{\Lambda^l \delta^l \left|\Psi(\theta_i, Z_i)\right|^2} d\theta_i\right) \right. \tag{254}$$
$$\left. \exp\left(\nabla_\theta \omega^{-1}\left(\theta, Z, |\Psi|^2\right) + \Psi^\dagger(\theta, Z)\nabla_\theta \omega^{-1}\left(\theta, Z, |\Psi|^2\right)\Psi(\theta, Z)\right)\right]_{\Psi(\theta,Z)=\Psi^\dagger(\theta,Z)=0}$$

4.3.2 Lowest order corrections

Given that $p_l + \sum_i p_l^i \geqslant 2$, the lowest order correction in (**??**) is for $m = 1, j = 1$, $p_l = p_1^1 = 1$:

$$\omega_e^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) \tag{255}$$
$$= \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$$
$$+\frac{1}{4}\int\int^{\theta} \frac{\delta\left(\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \times \frac{\delta\left(\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2$$

We rewrite the second term in the RHS of (255):

$$Z \equiv \int\int^{\theta} \frac{\delta\left(\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \frac{\delta\left(\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2$$

using an integration by part for the variable θ:

$$Z = \int \frac{\delta\left(\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \frac{\delta\left(\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2$$
$$-\int\int^{\theta} \frac{\delta\left(\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \frac{\delta\left(\nabla_\theta^2 \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2$$

We then use the local frequency equation (97) to rewrite $\nabla_{\theta}^{2}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)$ in the last term. In first approximation:

$$\nabla_{\theta}^{2}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\simeq\frac{f\Omega\left(\theta,Z\right)-\left(\frac{\hat{f}_{1}}{\omega(\theta,Z)}+N_{1}\right)\nabla_{\theta}\Omega\left(\theta,Z\right)-c^{2}\hat{f}_{3}\nabla_{Z}^{2}\Omega\left(\theta,Z\right)}{\omega_{0}^{2}\left(\theta,Z\right)\left(\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}\right)}$$

We will neglect the term $c^{2}\hat{f}_{3}\nabla_{Z}^{2}\Omega\left(\theta,Z\right)$ assuming that $\Omega\left(\theta,Z\right)$ varies slowly for a given time θ across space. Consequently, we have:

$$\begin{aligned}
Z &= \int\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{\delta\left(\nabla_{\theta}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
&+\int\int^{\theta}\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\\
&\times\frac{\delta}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{f\Omega\left(\theta,Z\right)-\left(\frac{\hat{f}_{1}}{\omega(\theta,Z)}+N_{1}\right)\nabla_{\theta}\Omega\left(\theta,Z\right)-c^{2}\hat{f}_{3}\nabla_{Z}^{2}\Omega\left(\theta,Z\right)}{\omega_{0}^{2}\left(\theta,Z\right)\left(\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}\right)}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}
\end{aligned}$$

Regrouping the terms in the previous expression yields:

$$\begin{aligned}
Z &\simeq \int\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{\delta\left(\nabla_{\theta}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
&+\frac{f}{\left(\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}\right)}\int\int^{\theta}\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{\delta\omega\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)}{\omega_{0}^{2}\left(\theta,Z\right)\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
&-\frac{\frac{\hat{f}_{1}}{\omega(\theta,Z)}+N_{1}}{\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}}\int\int^{\theta}\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\omega_{0}^{2}\left(\theta,Z\right)\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{\delta\nabla_{\theta}\omega\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
&\simeq \int\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{\delta\left(\nabla_{\theta}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
&-\frac{f}{\left(\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}\right)}\int\int^{\theta}\left(\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\right)^{2}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
&+\frac{\frac{\hat{f}_{1}}{\omega(\theta,Z)}+N_{1}}{\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}}\int\int^{\theta}\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{\delta\nabla_{\theta}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}
\end{aligned}$$

and this rewrites ultimately as:

$$\begin{aligned}
&\int\int^{\theta}\frac{\delta\left(\nabla_{\theta}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\frac{\delta\left(\nabla_{\theta}\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
\simeq\;&\int\frac{1}{2}\nabla_{\theta}\left(\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\right)^{2}\\
&-\frac{f}{\left(\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}\right)}\int\int^{\theta}\left(\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\right)^{2}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}\\
&+\frac{\frac{\hat{f}_{1}}{\omega(\theta,Z)}+N_{1}}{2\left(\frac{\hat{f}_{3}}{\omega(\theta,Z)}-N_{2}\right)}\int\left(\frac{\delta\left(\omega^{-1}\left(J\left(\theta\right),\theta,Z,\mathcal{G}_{0}+\left|\Psi\right|^{2}\right)\right)}{\delta\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}}\right)^{2}\left|\Psi\left(\theta^{(l)},Z_{l}\right)\right|^{2}
\end{aligned}$$

Which is the result stated in the text.

Appendix 5 Estimation of $\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$

To compute the effective action, the vacuum, the Green functions and to find non local solutions of (84) we will need to compute $\omega^{-1}(J,\theta,Z)$ and its derivatives $\frac{\delta^n\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n}\delta|\Psi(\theta-l_i,Z_i)|^2}$ appearing in (126). In Appendix 6 we will show that, in first approximation, the computation relies on the case $n=1$.

The first order derivatives $\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$ can be computed recursively. To do so, we will need to approximate the results around some static solution. We define $\bar{\omega}$ as solution of:

$$\begin{aligned}
\bar{\omega}^{-1}\left(J,Z\right) &= G\left(\bar{J}\left(Z\right)+\int\frac{\kappa}{N}\frac{\bar{\omega}\left(\bar{J},Z_{1}\right)}{\bar{\omega}\left(\bar{J},Z\right)}\mathcal{G}_{0}\left(0,0,Z_{1}\right)T\left(Z,Z_{1}\right)dZ_{1}\right) \qquad (256)\\
&= G\left(\bar{J}\left(Z\right)+\int\frac{\kappa}{N}\frac{\bar{\omega}\left(\bar{J},Z_{1}\right)}{\bar{\omega}\left(\bar{J},Z\right)\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^{2}\bar{X}_{r}}\right)^{2}+\frac{2\pi\alpha}{\sigma^{2}}}}T\left(Z,Z_{1}\right)dZ_{1}\right)
\end{aligned}$$

where $\bar{J}(Z)$ is the average of $J(\theta,Z)$ over the full timespan. We also define:

$$G_{0}'\left(J,Z\right)\equiv G'\left(\bar{J}\left(Z\right)+\int\frac{\kappa}{N}\frac{\bar{\omega}\left(\bar{J},Z_{1}\right)}{\bar{\omega}\left(J,\theta,Z\right)}G_{0Z_{1}}\left(0,0\right)T\left(Z,Z_{1}\right)dZ_{1}\right) \qquad (257)$$

These quantities will be useful below.

5.1 Computation of the first order derivatives in (126)

5.1.1 General formula

Using the recursive definition of $\omega^{-1}(J,\theta,Z)$:

$$\omega^{-1}\left(J,\theta,Z\right)=G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)}{\omega\left(J,\theta,Z\right)}\bar{W}\left(\frac{\omega\left(\theta,Z\right)}{\omega\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)}\right)\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}T\left(Z,Z_{1}\right)dZ_{1}\right) \qquad (258)$$

with:

$$\bar{W}\left(\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega\left(J,\theta,Z\right)}\right)=\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega\left(J,\theta,Z\right)}W\left(\frac{\omega\left(\theta,Z\right)}{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)$$

we first compute $\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$:

$$\begin{aligned}&\frac{\delta\omega^{-1}\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2}\\=&\frac{\delta G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\bar{W}\left(\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2T\left(Z,Z'\right)dZ'\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2}\end{aligned}\tag{259}$$

Expanding the right hand side and regrouping $\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$ on the left yields:

$$\begin{aligned}&\frac{\delta\omega^{-1}\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2}\\=&\frac{\frac{\kappa}{N}\bar{W}\left(\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)}\right)T\left(Z,Z_1\right)G'\left[J,\omega,\theta,Z,\Psi\right]\delta\left(l_1-\frac{|Z-Z_1|}{c}\right)}{1-\left(\int\frac{\kappa}{N}\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\bar{W}'\left(\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\omega(J,\theta,Z)}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2T\left(Z,Z'\right)dZ'\right)G'\left[J,\omega,\theta,Z,\Psi\right]}\\&+\frac{\frac{1}{\omega(J,\theta,Z)}\int\frac{\kappa}{N}\frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2}\bar{W}'\left(\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2T\left(Z,Z'\right)dZ'G'\left[J,\omega,\theta,Z,\Psi\right]}{1-\left(\int\frac{\kappa}{N}\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\bar{W}'\left(\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\omega(J,\theta,Z)}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2T\left(Z,Z'\right)dZ'\right)G'\left[J,\omega,\theta,Z,\Psi\right]}\\=&\omega\left(J,\theta-l_1,Z_1\right)\hat{T}_1\left(\theta,Z,Z_1,\omega,\Psi\right)\delta\left(l_1-\frac{|Z-Z_1|}{c}\right)\\&+\int\frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\hat{T}_1\left(\theta,Z,Z',\omega,\Psi\right)dZ'\end{aligned}\tag{260}$$

where we defined:

$$\begin{aligned}&\hat{T}_1\left(\theta,Z,Z_1,\omega,\Psi\right)=\frac{1}{\omega\left(J,\theta,Z\right)}\\&\times\frac{\frac{\kappa}{N}T\left(Z,Z_1\right)\bar{W}'\left(\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)}\right)G'\left[J,\omega,\theta,Z,\Psi\right]}{1-\left(\int\frac{\kappa}{N}\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\bar{W}'\left(\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\omega(J,\theta,Z)}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2T\left(Z,Z'\right)dZ'\right)G'\left[J,\omega,\theta,Z,\Psi\right]}\end{aligned}\tag{261}$$

Equation (259) shows that we also need $\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$ to compute $\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$. This is obtained by:

$$\begin{aligned}
\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &= \frac{\delta F\left(J(\theta,Z)+\int\frac{\kappa}{N}\bar{W}\left(\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\omega(J,\theta,Z)}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 T(Z,Z')\,dZ'\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\
&= \omega(J,\theta-l_1,Z_1)\,\hat{T}(\theta,Z,Z_1,\omega,\Psi)\,\delta\left(l_1-\frac{|Z-Z_1|}{c}\right) \\
&\quad+\int\frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\hat{T}(\theta,Z,Z',\omega,\Psi)\,dZ' \qquad (262)
\end{aligned}$$

with:

$$\begin{aligned}
&\hat{T}(\theta,Z,Z_1\omega,\Psi) \qquad (263)\\
&= \frac{\frac{\kappa}{N}\omega(J,\theta,Z)\,T(Z,Z_1)\,\bar{W}'\left(\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)}\right)F'[J,\omega,\theta,Z,\Psi]}{\omega^2(J,\theta,Z)+\left(\int\frac{\kappa}{N}\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\bar{W}'\left(\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\omega(J,\theta,Z)}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2T(Z,Z')\,dZ'\right)F'[J,\omega,\theta,Z,\Psi]}
\end{aligned}$$

Equation (262) and (263) define $\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}$ recursively. Actually, writing:

$$\begin{aligned}
&\frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\\
&= \int\omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right)\delta\left(\frac{|Z-Z'|}{c}+\frac{|Z'-Z''|}{c}-l_1\right)dZ''\\
&+\int\frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\right|^2\hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right)dZ''
\end{aligned}$$

we have:

$$\begin{aligned}
&\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\\
&= \int\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\hat{T}(\theta,Z,Z_1,\omega,\Psi)\,\delta\left(\frac{|Z-Z'|}{c}-l_1\right)dZ'\\
&+\int\omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right)\\
&\times\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\hat{T}(\theta,Z,Z',\omega,\Psi)\,\delta\left(\frac{|Z-Z'|}{c}+\frac{|Z'-Z''|}{c}-l_1\right)dZ'dZ''\\
&+\int\frac{\delta\omega\left(J,\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c}-\frac{|Z'-Z''|}{c},Z''\right)\right|^2\\
&\times\hat{T}\left(\theta-\frac{|Z-Z'|}{c},Z',Z'',\omega,\Psi\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\hat{T}(\theta,Z,Z',\omega,\Psi)\,dZ'dZ''
\end{aligned}$$

By a redefinition of $\hat{T}$ and $\hat{T}_1$:

$$\hat{T}(\theta, Z, Z', \omega, \Psi)\left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2 \rightarrow \hat{T}(\theta, Z, Z', \omega, \Psi)$$
$$\hat{T}_1(\theta, Z, Z', \omega, \Psi)\left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2 \rightarrow \hat{T}_1(\theta, Z, Z', \omega, \Psi)$$

which yields the series expansion:

$$\begin{aligned}\frac{\delta\omega(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2} &= \sum_{n=1}^{\infty}\frac{1}{\left|\Psi(\theta-l_1,Z_1)\right|^2}\int\omega\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right) \\ &\times\prod_{l=1}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}\end{aligned} \tag{264}$$

and:

$$\begin{aligned}\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2} &= \sum_{n=1}^{\infty}\frac{1}{\left|\Psi(\theta-l_1,Z_1)\right|^2}\int\omega\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)\hat{T}_1\left(\theta,Z,Z^{(1)},\omega,\Psi\right) \\ &\times\prod_{l=2}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}\end{aligned} \tag{265}$$

with the convention that $Z^{(0)}=Z$ and $Z^{(n)}=Z_1$.

We can write (265) in a more symetric way. Defining:

$$\check{T}\left(\theta,Z,Z^{(1)},\omega,\Psi\right) = -\omega^2(J,\theta,Z)\,\hat{T}_1\left(\theta,Z,Z^{(1)},\omega,\Psi\right)$$

Relation (260) writes:

$$\begin{aligned}\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2} &= \omega^{-1}(J,\theta-l_1,Z_1)\,\check{T}(\theta,Z,Z_1,\omega,\Psi)\,\delta\left(l_1-\frac{|Z-Z_1|}{c}\right) \\ &+\int\frac{\delta\omega^{-1}\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\check{T}(\theta,Z,Z',\omega,\Psi)\,dZ'\end{aligned}$$

and we have:

$$\begin{aligned}\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2} &= \sum_{n=1}^{\infty}\frac{1}{\left|\Psi(\theta-l_1,Z_1)\right|^2}\int\omega^{-1}\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right) \\ &\times\prod_{l=1}^{n}\check{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}\end{aligned} \tag{266}$$

5.1.2 Static approximation

We now use the static approximations (256) and (257). Actually, the values of $\hat{T}_1(\theta,Z,Z_1\omega,\Psi)$ and $\hat{T}(\theta,Z,Z_1\omega,\Psi)$ can be estimated for $\bar{\omega}^{-1}\left(\bar{J},Z\right)$. Moreover, in the limit of small fluctuations, $\bar{\omega}^{-1}\left(\bar{J},Z\right)$, $F'[J,\bar{\omega},Z,\Psi]$ and $G'[J,\bar{\omega},Z,\Psi]$ can be approximated by their average over Z, denoted $\bar{\omega}^{-1}$, $\bar{F}'$and $\bar{G}'$. We also have:

$$\frac{\bar{\omega}(J,Z')}{\bar{\omega}(J,Z)} \simeq 1$$

We also replace $|\Psi|^2$ by $\frac{1}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\pi\alpha}{\sigma^2}}}$. Moreover for $\bar{\omega}$, both $\hat{T}_1$ and $\hat{T}$ can be considered independent of θ:

$$\begin{aligned}
\hat{T}_1(\theta, Z, Z_1\bar{\omega}, \Psi) &\simeq \hat{T}_1(Z, Z_1, \bar{\omega}) \\
&= \frac{1}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\pi\alpha}{\sigma^2}}} \frac{\frac{\kappa}{N}\bar{\omega}^{-1}T(Z, Z_1)\bar{G}'}{1-\frac{\bar{G}'\bar{\omega}\int\frac{\kappa}{N}T(Z,Z')dZ'}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\pi\alpha}{\sigma^2}}}}
\end{aligned}$$

$$\begin{aligned}
\hat{T}(\theta, Z, Z_1\omega, \Psi) &\simeq \hat{T}(Z, Z_1, \bar{\omega}) \\
&= \frac{1}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\pi\alpha}{\sigma^2}}} \frac{\frac{\kappa}{N}T(Z, Z_1)\bar{F}'}{\bar{\omega}+\frac{\bar{F}'\int\frac{\kappa}{N}T(Z,Z')dZ'}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\pi\alpha}{\sigma^2}}}}
\end{aligned}$$

as a consequence $\hat{T}_1(Z, Z_1, \bar{\omega})$ and $\hat{T}(Z, Z_1, \bar{\omega})$ are functions of $|Z - Z_1|$ denoted $\hat{T}_1(|Z - Z_1|)$. As a consequence (265) can be estimated by:

$$\begin{aligned}
\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &= \sum_{n=1}^{\infty}\frac{1}{|\Psi(\theta-l_1,Z_1)|^2}\int\omega(J,\theta-l_1,Z_1)\hat{T}_1\left(\left|Z-Z^{(1)}\right|\right) \\
&\times\prod_{l=2}^{n}\hat{T}\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right) \\
&\times\delta\left(Z-Z_1-\sum_{l=1}^{n}\left(Z^{(l-1)}-Z^{(l)}\right)\right)\prod_{l=1}^{n-1}dZ^{(l)}
\end{aligned} \tag{267}$$

and (264) is:

$$\begin{aligned}
\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &= \sum_{n=1}^{\infty}\frac{1}{|\Psi(\theta-l_1,Z_1)|^2}\int\omega\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)\prod_{l=1}^{n}\hat{T}\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right) \\
&\times\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\times\delta\left(Z-Z_1-\sum_{l=1}^{n}\left(Z^{(l-1)}-Z^{(l)}\right)\right)\prod_{l=1}^{n-1}dZ^{(l)}
\end{aligned} \tag{268}$$

5.2 Estimation of (267) and (264) close to the permanent regime

The series (267) can be computed by using the Fourier transform of the Dirac functions:

$$\begin{aligned}
|\Psi(\theta-l_1,Z_1)|^2\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &= \sum_{n=1}^{\infty}\int\omega(J,\theta-l_1,Z_1)\times\hat{T}_1\left(\left|Z-Z^{(1)}\right|\right) \\
&\times\prod_{l=2}^{n}\hat{T}\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)\exp\left(i\lambda\left(cl_1-\sum_{l=1}^{n}\left|Z^{(l-1)}-Z^{(l)}\right|\right)\right) \\
&\times\exp\left(i\lambda_1.\left(Z-Z_1-\sum_{l=1}^{n}\left(Z^{(l-1)}-Z^{(l)}\right)\right)\right)d\lambda d\lambda_1 \\
&\times\prod_{l=1}^{n}\left|Z^{(l-1)}-Z^{(l)}\right|^2 d\left|Z^{(l-1)}-Z^{(l)}\right|dv_l
\end{aligned} \tag{269}$$

where the unit vectors v_l are defined such that:

$$Z^{(l-1)}-Z^{(l)}=v_l\left|Z^{(l-1)}-Z^{(l)}\right|$$

We also define:

$$\begin{aligned}\lambda_1.(Z - Z_1) &= |\lambda_1| |Z - Z_1| \cos(\theta_1) \\ \lambda_1.v_l &= |\lambda_1| \cos(\theta_l)\end{aligned}$$

The angles θ_l are computed in the plane $(\lambda_1, Z - Z_1)$ between the projection of v_l and $Z - Z_1$.

Before computing the integrals in (269) for arbitrary transfer functions, we develop the particular case of an exponential transfer function.

5.2.1 Exponential transfer function

We first choose:

$$\begin{aligned}\hat{T}\left(\left|Z^{(l-1)} - Z^{(l)}\right|\right) &= C\frac{\exp\left(-c\left|Z^{(l-1)} - Z^{(l)}\right|\right)}{\left|Z^{(l-1)} - Z^{(l)}\right|} \\ \hat{T}\left(\left|Z^{(l-1)} - Z^{(l)}\right|\right) &\simeq \frac{A_1}{A}\hat{T}\left(\left|Z^{(l-1)} - Z^{(l)}\right|\right)\end{aligned} \tag{270}$$

and discard the factor $\frac{A_1}{A}$ that will be reintroduced in the end of the computation.

Using that $\sum_{l=1}^{n}\left(Z^{(l-1)} - Z^{(l)}\right) = cl_1$, the right hand side of (269) becomes:

$$\begin{aligned}&\exp(-cl_1) \times \sum_{n=1}^{\infty} \int \exp\left(i\lambda\left(cl_1 - \sum_{l=1}^{n}\left|Z^{(l-1)} - Z^{(l)}\right|\right)\right) \\ &\times \exp\left(i\lambda_1.\left(Z - Z_1 - \sum_{l=1}^{n}\left(Z^{(l-1)} - Z^{(l)}\right)\right)\right) d\lambda d\lambda_1 \prod_{l=1}^{n} C\left|Z^{(l-1)} - Z^{(l)}\right| d\left|Z^{(l-1)} - Z^{(l)}\right| dv_l\end{aligned}$$

that can be written in terms of the angles as:

$$\begin{aligned}&\exp(-cl_1) \times \sum_{n=1}^{\infty} \int \exp\left(i\lambda cl_1 + i|\lambda_1||Z - Z_1|\cos(\theta_1)\right) \\ &\times \exp\left(-i\sum_{l=1}^{n}\left(\lambda + |\lambda_1|\cos(\theta_l)\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right) d\lambda d\lambda_1 \prod_{l=1}^{n} C\left|Z^{(l-1)} - Z^{(l)}\right| d\left|Z^{(l-1)} - Z^{(l)}\right| dv_l\end{aligned} \tag{271}$$

The integration over θ_l is:

$$\begin{aligned}&\pi\int_0^{\pi} \exp\left(-i\left(\lambda + |\lambda_1|\cos(\theta_l)\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right)\sin(\theta_l)\,d\theta_l \\ = &-\frac{\pi i}{|\lambda_1|\left|Z^{(l-1)} - Z^{(l)}\right|}\left(\exp\left(-i\left(\lambda - |\lambda_1|\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right) - \exp\left(-i\left(\lambda + |\lambda_1|\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right)\right) \\ = &\frac{\pi i}{|\lambda_1|\left|Z^{(l-1)} - Z^{(l)}\right|}\left(\exp\left(-i\left(\lambda + |\lambda_1|\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right) - \exp\left(-i\left(\lambda - |\lambda_1|\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right)\right)\end{aligned}$$

and (271) rewrites:

$$\begin{aligned}&\exp(-cl_1) \times \sum_{n=1}^{\infty} \int \frac{-\pi i}{|\lambda_1||Z - Z_1|}\left(\exp\left(i\lambda cl_1 + i|\lambda_1||Z - Z_1|\right) - \exp\left(i\lambda cl_1 - i|\lambda_1||Z - Z_1|\right)\right) \\ &\times \prod_{l=1}^{n} C\frac{\pi i}{|\lambda_1|}\left(\exp\left(-i\left(\lambda + |\lambda_1|\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right) - \exp\left(-i\left(\lambda - |\lambda_1|\right)\left|Z^{(l-1)} - Z^{(l)}\right|\right)\right) d\left|Z^{(l-1)} - Z^{(l)}\right| d\lambda\,|\lambda_1|^2\,d|\lambda_1|\end{aligned}$$

We can then perform the integrals over the norms $\left|Z^{(l-1)} - Z^{(l)}\right|$, which yields:

$$\exp(-cl_1)\times\sum_{n=1}^{\infty}\int\frac{-\pi i}{|\lambda_1|\,|Z-Z_1|}\left(\exp\left(i\lambda cl_1+i\,|\lambda_1|\,|Z-Z_1|\right)-\exp\left(i\lambda cl_1-i\,|\lambda_1|\,|Z-Z_1|\right)\right)$$
$$\times\prod_{l=1}^{n}C\frac{\pi}{|\lambda_1|}\left(\frac{1}{\lambda+|\lambda_1|-i\varepsilon}-\frac{1}{\lambda-|\lambda_1|-i\varepsilon}\right)d\lambda\,|\lambda_1|^2\,d\,|\lambda_1|$$

Performing the sum yields then the following expression for (271):

$$\begin{aligned}
&\exp(-cl_1)\times\int\frac{-\pi i}{|\lambda_1|\,|Z-Z_1|}\left(\exp\left(i\lambda cl_1+i\,|\lambda_1|\,|Z-Z_1|\right)-\exp\left(i\lambda cl_1-i\,|\lambda_1|\,|Z-Z_1|\right)\right)\\
&\times\frac{-C\frac{2\pi}{(\lambda+|\lambda_1|-i\varepsilon)(\lambda-|\lambda_1|-i\varepsilon)}}{1+C\frac{2\pi}{(\lambda+|\lambda_1|-i\varepsilon)(\lambda-|\lambda_1|-i\varepsilon)}}d\lambda\,|\lambda_1|^2\,d\,|\lambda_1|\\
=\;&\exp(-cl_1)\times\int\frac{-\pi i}{|\lambda_1|\,|Z-Z_1|}\left(\exp\left(i\lambda cl_1+i\,|\lambda_1|\,|Z-Z_1|\right)-\exp\left(i\lambda cl_1-i\,|\lambda_1|\,|Z-Z_1|\right)\right)\\
&\times\frac{-2\pi C}{(\lambda+|\lambda_1|-i\varepsilon)(\lambda-|\lambda_1|-i\varepsilon)+2\pi C}d\lambda\,|\lambda_1|^2\,d\,|\lambda_1|
\end{aligned}$$

Ultimately, the previous formula can be reduced to a single expression, by performing the change of variable $x=-|\lambda_1|$ in the term with $\exp\left(i\lambda cl_1-i\,|\lambda_1|\,|Z-Z_1|\right)$ in factor. We obtain:

$$\exp(-cl_1)\times\int\frac{-\pi i}{|Z-Z_1|}\exp\left(i\lambda cl_1+i\lambda_1\,|Z-Z_1|\right)\frac{-2\pi C\lambda_1}{(\lambda+\lambda_1-i\varepsilon)(\lambda-\lambda_1-i\varepsilon)+2\pi C}d\lambda d\lambda_1$$

where the integral over λ_1 is now performed with $\lambda_1\in\mathbb{R}$. This integral is computed by the residue theorem, where the residues satisfy:

$$\lambda_1^2=(\lambda-i\varepsilon)^2+2\pi C$$

leading to write (271) as:

$$\begin{aligned}
&\exp(-cl_1)\times\int\frac{-\pi i}{|Z-Z_1|}\exp\left(i\lambda cl_1+i\sqrt{(\lambda-i\varepsilon)^2+2\pi C}\,|Z-Z_1|\right)d\lambda\\
&+\exp(-cl_1)\times\int\frac{-\pi i}{|Z-Z_1|}\exp\left(i\lambda cl_1-i\sqrt{(\lambda-i\varepsilon)^2+2\pi C}\,|Z-Z_1|\right)d\lambda
\end{aligned}\tag{272}$$

We then perform the change of variable:

$$\begin{aligned}
x&=\lambda+\sqrt{\lambda^2+2\pi C}\\
dx&=\left(1+\frac{\lambda}{\sqrt{\lambda^2+2\pi C}}\right)d\lambda\\
&=\frac{x}{\sqrt{\lambda^2+2\pi C}}d\lambda=\frac{2x^2}{x^2+2\pi C}d\lambda
\end{aligned}$$

and rewrite the exponents in (272) as:

$$\begin{aligned}
\lambda cl_1+\sqrt{(\lambda-i\varepsilon)^2+2\pi C}\,|Z-Z_1|&=\frac{cl_1+|Z-Z_1|}{2}\left(\lambda+\sqrt{\lambda^2+2\pi C}\right)\\
&\quad+\frac{cl_1-|Z-Z_1|}{2}\left(\lambda-\sqrt{\lambda^2+2\pi C}\right)\\
&=\frac{cl_1+|Z-Z_1|}{2}\left(\lambda+\sqrt{\lambda^2+2\pi C}\right)-\frac{cl_1-|Z-Z_1|}{2}\frac{2\pi C}{\lambda+\sqrt{\lambda^2+2\pi C}}\\
&=\frac{cl_1+|Z-Z_1|}{2}x-\frac{cl_1-|Z-Z_1|}{2}\frac{2\pi C}{x}
\end{aligned}$$

and:

$$\lambda cl_1 - \sqrt{(\lambda - i\varepsilon)^2 + 2\pi C}\,|Z - Z_1| = \frac{cl_1 - |Z - Z_1|}{2}x - \frac{cl_1 + |Z - Z_1|}{2}\frac{2\pi C}{x}$$

As a consequence, expression (272) becomes:

$$\begin{aligned}
&\exp(-cl_1) \times \int \frac{-\pi i}{|Z - Z_1|} \exp\left(i\left(\frac{cl_1 + |Z - Z_1|}{2}x - \frac{cl_1 - |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) dx \\
&+ \exp(-cl_1) \times \int \frac{-\pi i}{|Z - Z_1|} \exp\left(i\left(\frac{cl_1 - |Z - Z_1|}{2}x - \frac{cl_1 + |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) dx \\
&+ 2\pi C \exp(-cl_1) \times \int \frac{-\pi i}{|Z - Z_1|} \exp\left(i\left(\frac{cl_1 + |Z - Z_1|}{2}x - \frac{cl_1 - |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) \frac{1}{x^2} dx \\
&+ 2\pi C \times \int \frac{-\pi i}{|Z - Z_1|} \exp\left(i\left(\frac{cl_1 - |Z - Z_1|}{2}x - \frac{cl_1 + |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) \frac{1}{x^2} dx
\end{aligned}$$

Performing the change of variable $y = \frac{1}{x}$ in the two last expressions yields:

$$\begin{aligned}
&\exp(-cl_1)(1 + 2\pi C) \times \left(\int \frac{-\pi i}{|Z - Z_1|} \exp\left(i\left(\frac{cl_1 + |Z - Z_1|}{2}x - \frac{cl_1 - |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) dx\right. \\
&\left.+ \int \frac{-\pi i}{|Z - Z_1|} \exp\left(i\left(\frac{cl_1 - |Z - Z_1|}{2}x - \frac{cl_1 + |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) dx\right)
\end{aligned}$$

and by analytic continuation $x \to ix$, this becomes:

$$\begin{aligned}
&\exp(-cl_1)(1 + 2\pi C) \times \left(\int \frac{\pi}{|Z - Z_1|} \exp\left(-\left(\frac{cl_1 + |Z - Z_1|}{2}x - \frac{cl_1 - |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) dx\right. \\
&\left.+ \int \frac{\pi}{|Z - Z_1|} \exp\left(-\left(\frac{cl_1 - |Z - Z_1|}{2}x - \frac{cl_1 + |Z - Z_1|}{2}\frac{2\pi C}{x}\right)\right) dx\right)
\end{aligned}$$

Ultimately, reintroducing the constraint $H(cl_1 - |Z - Z_1|)$ and the factor $\frac{A_1}{A}$, (269) writes:

$$\begin{aligned}
|\Psi(\theta - l_1, Z_1)|^2 \frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &= (1 + 2\pi C)\frac{A_1}{A}\frac{\exp(-cl_1)}{|Z - Z_1|}\left(\sqrt{\frac{cl_1 - |Z - Z_1|}{cl_1 + |Z - Z_1|}} + \sqrt{\frac{cl_1 + |Z - Z_1|}{cl_1 - |Z - Z_1|}}\right) \\
&\quad \times K_1\left(\frac{cl_1 - |Z - Z_1|}{2}2\pi C\frac{cl_1 + |Z - Z_1|}{2}\right)\omega(J, \theta - l_1, Z_1) \\
&= (1 + 2\pi C)\frac{A_1}{A}\frac{\exp(-cl_1)}{|Z - Z_1|}\left(\sqrt{\frac{cl_1 - |Z - Z_1|}{cl_1 + |Z - Z_1|}} + \sqrt{\frac{cl_1 + |Z - Z_1|}{cl_1 - |Z - Z_1|}}\right) \\
&\quad \times K_1\left(\pi C\frac{(cl_1)^2 - |Z - Z_1|^2}{2}\right)\omega(J, \theta - l_1, Z_1) \qquad (273)
\end{aligned}$$

In first approximation, the right hand side of (273) is:

$$\begin{aligned}
&\frac{\exp(-cl_1)(cl_1 + |Z - Z_1|)}{B|Z - Z_1|}\exp\left(-\pi C\frac{(cl_1)^2 - |Z - Z_1|^2}{2}\right)\omega(J, \theta - l_1, Z_1) \qquad (274) \\
\sim\ &\frac{\exp(-cl_1)}{B}\exp\left(-\pi C cl_1\frac{cl_1 - |Z - Z_1|}{2}\right)H(cl_1 - |Z - Z_1|)\,\omega(J, \theta - l_1, Z_1)
\end{aligned}$$

for $cl_1 >> |Z - Z_1|$. This can also be replaced by a simplest form:

$$|\Psi(\theta - l_1, Z_1)|^2 \frac{\delta\omega^{-1}(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} \simeq \frac{\exp\left(-cl_1 - \alpha\left((cl_1)^2 - |Z - Z_1|^2\right)\right)}{B}H(cl_1 - |Z - Z_1|)\,\omega(J, \theta - l_1, Z_1) \qquad (275)$$

where B and α are constants.

Using (266), the same computation can be performed by replacing $\hat{T}$ with $\check{T}$ and we obtain:

$$\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2\frac{\delta\omega^{-1}\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2}\simeq\frac{\exp\left(-cl_1-\alpha\left(\left(cl_1\right)^2-\left|Z-Z_1\right|^2\right)\right)}{D}H\left(cl_1-\left|Z-Z_1\right|\right)\omega^{-1}\left(J,\theta-l_1,Z_1\right) \tag{276}$$

with D a constant.

5.2.2 General formula

For an arbitrary transfer function:

$$\hat{T}\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)=C\exp\left(-c\left|Z^{(l-1)}-Z^{(l)}\right|\right)f\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)$$

we can factor $C\exp\left(-cl\right)$ as in the previous paragraph. It amounts to replace:

$$\hat{T}\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)\rightarrow f\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)$$

We rewrite (269) as:

$$\begin{aligned}
&\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2\frac{\delta\omega^{-1}\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2} \qquad (277)\\
=&\sum_{n=1}^{\infty}\int\omega\left(J,\theta-l_1,Z_1\right)\times T_1''\left(\lambda+\lambda_1.v_1\right)dv_1\prod_{l=2}^{n}\int T''\left(\lambda+\lambda_1.v_l\right)dv_l\exp\left(i\lambda cl_1+i\lambda_1.\left(Z-Z_1\right)\right)d\lambda d\lambda_1\\
=&\,\delta\left(\left|Z_1-Z\right|-cl_1\right)\hat{T}_1\left(\left|Z-Z^{(1)}\right|\right)\omega\left(J,\theta-l_1,Z_1\right)\\
&+\left(-1\right)^n\int\omega\left(J,\theta-l_1,Z_1\right)\times\frac{T_1''\left(\lambda+\lambda_1.v_1\right)}{2}dv_1\prod_{l=2}^{n}\int\frac{T''\left(\lambda+\lambda_1.v_l\right)}{2}dv_l\exp\left(i\lambda cl_1+i\lambda_1.\left(Z-Z_1\right)\right)d\lambda d\lambda_1
\end{aligned}$$

With the convention that for $n=1$, the product $\prod_{l=2}^{n}$ is set to be equal to 1. The functions T_1 and T are the fourier transform of $\hat{T}_1H$ and $\hat{T}H$ respectively, and H is the heaviside function. Remark that the first term of (277) expresses the Dirac function $\delta\left(\left|Z_1-Z\right|-cl_1\right)$ as a Fourier transform:

$$\begin{aligned}
&\exp\left(i\lambda\left(cl_1-\sum_{l=1}^{n}\left|Z^{(0)}-Z^{(1)}\right|\right)\right)\\
&\times\exp\left(i\lambda_1.\left(Z-Z_1-\sum_{l=1}^{n}\left(Z^{(0)}-Z^{(1)}\right)\right)\right)d\lambda d\lambda_1\left|Z^{(0)}-Z^{(1)}\right|^2d\left|Z^{(0)}-Z^{(1)}\right|dv_l
\end{aligned}$$

Some terms of (277) can be written in a useful form for the sequel:

$$\begin{aligned}
\frac{1}{2}\int T''\left(\lambda+\lambda_1.v_l\right)dv_l &= \pi\int_0^{\pi}T''\left(\lambda+\left|\lambda_1\right|\cos\left(\theta_l\right)\right)\sin\left(\theta_l\right)d\theta_l\\
&= \pi\int_{-1}^{1}T''\left(\lambda+\left|\lambda_1\right|u\right)du\\
&= \frac{2\pi\left(T'\left(\lambda+\left|\lambda_1\right|\right)-T'\left(\lambda-\left|\lambda_1\right|\right)\right)}{2\left|\lambda_1\right|}\\
&\equiv \bar{T}\left(\lambda,\left|\lambda_1\right|\right)
\end{aligned} \tag{278}$$

$$\int T_1''(\lambda+\lambda_1.v_l)\,dv_l = \frac{2\pi\left(T_1'(\lambda+|\lambda_1|)-T_1'(\lambda-|\lambda_1|)\right)}{2|\lambda_1|} \equiv \bar{T}_1(\lambda,|\lambda_1|) \tag{279}$$

$$\begin{aligned}\exp(i\lambda_1.(Z-Z_1))\,d\lambda_1 &= \exp\left(i\cos(\theta_1)|\lambda_1||Z-Z_1|\right)\sin(\theta_1)|\lambda_1|^2 d|\lambda_1|d\theta_1 \\ &= \exp\left(iu|\lambda_1||Z-Z_1|\right)|\lambda_1|^2 d|\lambda_1|du\end{aligned} \tag{280}$$

Remark that the functions of x:

$$\bar{T}(\lambda,x)=\frac{2\pi\left(T'(\lambda+x)-T'(\lambda-x)\right)}{2x} \text{ and } \bar{T}_1(\lambda,x)=\frac{2\pi\left(T_1'(\lambda+x)-T_1'(\lambda-x)\right)}{2x}$$

are even.

5.2.3 Estimation of (277)

Using (278), (279) and (280), equation (277) becomes:

$$\begin{aligned}&|\Psi(\theta-l_1,Z_1)|^2\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\ =& \sum_{n=1}^{\infty}(-1)^n\int\omega(J,\theta-l_1,Z_1)\times T_1(\lambda+\lambda_1.v_1)\,dv_1\prod_{l=2}^{n}\int T(\lambda+\lambda_1.v_l)\,dv_l\exp\left(i\lambda cl_1+i\lambda_1.(Z-Z_1)\right)d\lambda d\lambda_1 \\ =& -\int\omega(J,\theta-l_1,Z_1)\times\frac{\bar{T}_1(\lambda,|\lambda_1|)}{1+\bar{T}(\lambda,|\lambda_1|)}\exp(i\lambda cl_1)\int_{-1}^{1}\exp\left(iu|\lambda_1||Z-Z_1|\right)|\lambda_1|^2 d|\lambda_1|dud\lambda \\ =& -\int\omega(J,\theta-l_1,Z_1)\times\frac{\bar{T}_1(\lambda,|\lambda_1|)}{1+\bar{T}(\lambda,|\lambda_1|)}\exp(i\lambda cl_1)\left(2\frac{\sin(|\lambda_1||Z-Z_1|)}{|Z-Z_1|}|\lambda_1|\right)d|\lambda_1|d\lambda\end{aligned} \tag{281}$$

We remark that for even functions f, the following identity holds:

$$\begin{aligned}&\int_0^{+\infty}f(|\lambda_1|)\,2\frac{\sin(|\lambda_1||Z-Z_1|)}{|Z-Z_1|}|\lambda_1|\,d|\lambda_1| \\ =& \int_0^{+\infty}f(x)\frac{\exp(ix|Z-Z_1|)-\exp(-ix|Z-Z_1|)}{i|Z-Z_1|}xdx \\ =& \int_0^{+\infty}f(x)\frac{\exp(ix|Z-Z_1|)}{i|Z-Z_1|}xdx+\int_{-\infty}^{0}f(-x)\frac{\exp(ix|Z-Z_1|)}{i|Z-Z_1|}xdx \\ =& \int_{-\infty}^{+\infty}f(x)\frac{\exp(ix|Z-Z_1|)}{i|Z-Z_1|}xdx\end{aligned}$$

so that (281) becomes:

$$\begin{aligned}&|\Psi(\theta-l_1,Z_1)|^2\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \\ =& -\int\omega(J,\theta-l_1,Z_1)\times\frac{\bar{T}_1(\lambda,\lambda_1)}{1+\bar{T}(\lambda,\lambda_1)}\frac{\lambda_1}{i|Z-Z_1|}\exp\left(i\lambda cl_1+i\lambda_1|Z-Z_1|\right)d\lambda_1 d\lambda \\ =& -\int\omega(J,\theta-l_1,Z_1)\times\frac{\pi\left(T_1'(\lambda+\lambda_1)-T_1'(\lambda-\lambda_1)\right)}{\lambda_1+\pi\left(T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)\right)}\frac{\lambda_1}{i|Z-Z_1|} \\ &\times\exp\left(i\lambda cl_1+i\lambda_1|Z-Z_1|\right)d\lambda_1 d\lambda \\ =& -\int\omega(J,\theta-l_1,Z_1)\times\frac{\pi\left(T_1'(\lambda+\lambda_1)-T_1'(\lambda-\lambda_1)\right)}{\lambda_1+\pi\left(T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)\right)}\frac{\lambda_1}{i|Z-Z_1|} \\ &\times\exp\left(iu\left(cl_1+|Z-Z_1|\right)\right)\times\exp\left(iv\left(cl_1-|Z-Z_1|\right)\right)d\lambda_1 d\lambda\end{aligned} \tag{282}$$

As in the previous paragraph, we also simplify (282) by writing T_1 as a function of T:

$$T_1'(\lambda+\lambda_1)-T_1'(\lambda-\lambda_1)=\frac{A_1}{A}\left(T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)\right)$$

and by setting:

$$\begin{aligned} u &= \frac{\lambda+\lambda_1}{2} \\ v &= \frac{\lambda+\lambda_1}{2} \end{aligned}$$

so that we are lead to:

$$\begin{aligned} |\Psi(\theta-l_1,Z_1)|^2\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &= -\int\omega(J,\theta-l_1,Z_1)\times\frac{A_1}{A}\frac{\pi\left(T_1'(\lambda+\lambda_1)-T_1'(\lambda-\lambda_1)\right)}{\lambda_1+\pi\left(T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)\right)}\frac{\lambda_1}{i|Z-Z_1|} \\ &\times\exp\left(iu\left(cl_1+|Z-Z_1|\right)\right)\times\exp\left(iv\left(cl_1-|Z-Z_1|\right)\right)d\lambda_1 d\lambda \qquad (283) \end{aligned}$$

Remark that the particular case of the exponential transfer function is encompassed in (282). Actually, if we choose:

$$\hat{T}\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)=C\frac{\exp\left(-c\left|Z^{(l-1)}-Z^{(l)}\right|\right)}{\left|Z^{(l-1)}-Z^{(l)}\right|}$$

we have:

$$\prod_{l=1}^{n}\hat{T}\left(\left|Z^{(l-1)}-Z^{(l)}\right|\right)=\exp(-cl_1)\prod_{l=1}^{n}\frac{C}{\left|Z^{(l-1)}-Z^{(l)}\right|}$$

For such a choice, we have formally: $T=-iC\int(FH)$ where H is the heaviside function. As a consequence:

$$T'(\lambda)=CFH=-\frac{C}{\lambda+i\varepsilon}$$

and (283) is equivalent to the expressions of appendix 1.3.2.1.

In the general case, we write $\lambda_1^{(r)}$, $r=1,...$ the solutions to the pole equation of (283):

$$\lambda_1+\pi\left(T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)\right)=0$$

For regular functions $T'(\lambda+\lambda_1)$ such that for $\lambda\to\infty$:

$$T'(\lambda+\lambda_1)\simeq\frac{g(\lambda+\lambda_1)}{(\lambda+\lambda_1)^l}$$

$$\int\frac{1}{(\lambda-s)^l}|\Psi(s)|$$

with $l>0$ given and g bounded, the poles equation implies that for $\lambda\to\infty$:

$$\lambda_1\simeq\pm\lambda$$

and as a consequence, we can write:

$$\lambda_1^{(r)}=\sqrt{\lambda^2+h_r(\lambda)} \qquad (284)$$

where $h_r(\lambda)$ is bounded.

We can compute the values of the residues at each pole by the first order expansion of $1+\pi\frac{T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)}{\lambda_1}$:

$$\begin{aligned}
&1+\pi\frac{T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)}{\lambda_1}\\
\simeq\ &\pi\left(\frac{T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)}{\lambda_1}-\frac{T'\left(\lambda+\lambda_1^{(r)}\right)-T'\left(\lambda-\lambda_1^{(r)}\right)}{\lambda_1^{(r)}}\right)\\
\simeq\ &\pi\left(\frac{\frac{1}{\pi}+T''\left(\lambda+\lambda_1^{(r)}\right)+T''\left(\lambda-\lambda_1^{(r)}\right)}{\lambda_1^{(r)}}\right)\\
\simeq\ &\pi\left(\frac{T''\left(\lambda+\lambda_1^{(r)}\right)+T''\left(\lambda-\lambda_1^{(r)}\right)-\frac{T'\left(\lambda+\lambda_1^{(r)}\right)-T'\left(\lambda-\lambda_1^{(r)}\right)}{\lambda_1^{(r)}}}{\lambda_1^{(r)}}\right)
\end{aligned}$$

For regular functions $T'(\lambda+\lambda_1)$, this can be expanded as:

$$2\pi\lambda_1^{(r)}\left(\sum_{k\geqslant 1}\frac{T^{(2k+2)}(\lambda)}{(2k)!}\left(\lambda_1^{(r)}\right)^{2k-2}-\sum_{k\geqslant 1}\frac{T^{(2k+2)}(\lambda)}{(2k+1)!}\left(\lambda_1^{(r)}\right)^{2k-2}\right)$$

and for relatively slowly varying functions, this reduces to:

$$1+\pi\frac{T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)}{\lambda_1}\simeq 2\pi\lambda_1^{(r)}\frac{T^{(4)}(\lambda)}{3} \tag{285}$$

and the residue theorem implies to replace:

$$\begin{aligned}
&\frac{\pi\left(T_1'(\lambda+\lambda_1)-T_1'(\lambda-\lambda_1)\right)}{\lambda_1+\pi\left(T'(\lambda+\lambda_1)-T'(\lambda-\lambda_1)\right)}\frac{\lambda_1}{i\,|Z-Z_1|}\\
\rightarrow\ &-\frac{i}{\pi}\frac{3}{|Z-Z_1|\,T^{(4)}(\lambda)}
\end{aligned} \tag{286}$$

in (283). Using (284) and (286) in (283) leads to:

$$\begin{aligned}
|\Psi(\theta-l_1,Z_1)|^2\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &\simeq \sum_r\frac{i}{\pi}\frac{1}{|Z-Z_1|}\int\omega(J,\theta-l_1,Z_1)\\
&\times\frac{3}{T^{(4)}(\lambda)}\exp\left(iu\left(cl_1+|Z-Z_1|\right)\right)\times\exp\left(iv\left(cl_1-|Z-Z_1|\right)\right)d\lambda\\
&= \sum_r\frac{i}{\pi}\frac{1}{|Z-Z_1|}\int\omega(J,\theta-l_1,Z_1)\\
&\times\frac{3}{T^{(4)}(\lambda)}\exp\left(iu\left(cl_1+|Z-Z_1|\right)\right)\times\exp\left(iv\left(cl_1-|Z-Z_1|\right)\right)d\lambda
\end{aligned}$$

where:

$$\begin{aligned}
u &= \frac{\lambda+\lambda_1^{(r)}}{2}=\frac{\lambda+f^{(r)}(\lambda)}{2}\\
v &= \frac{\lambda+\lambda_1^{(r)}}{2}=\frac{\lambda-f^{(r)}(\lambda)}{2}
\end{aligned}$$

As a consequence:

$$\begin{aligned}
v &= \lambda-\sqrt{\lambda^2+h_r(\lambda)}\\
&= -\frac{h_r(\lambda)}{\lambda+\sqrt{\lambda^2+h_r(\lambda)}}\\
&= -\frac{h_r(\lambda)}{u}
\end{aligned}$$

For $h_r(\lambda)$ varying slowly, we can replace $h_r(\lambda)$ by its average $\bar{h}_r$, and we have:

$$v = -\frac{\bar{h}_r}{u}$$

Replacing $T^{(4)}(\lambda)$ by its average $\bar{T}^{(4)}$, we find:

$$\begin{aligned}|\Psi(\theta - l_1, Z_1)|^2 \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &\simeq \sum_r \frac{i}{\pi}\frac{1}{|Z - Z_1|}\int \omega(J,\theta - l_1, Z_1) \\ &\times \frac{3}{\bar{T}^{(4)}}\exp(iu(cl_1 + |Z - Z_1|)) \times \exp\left(-i\frac{\bar{h}_r}{u}(cl_1 - |Z - Z_1|)\right)d\lambda\end{aligned}$$

We can then apply the results of the previous paragraph for each r, and has a consequence, we obtain:

$$\begin{aligned}|\Psi(\theta - l_1, Z_1)|^2 \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &\simeq \sum_r \left(1 + \bar{h}_r\right)\frac{3}{\bar{T}^{(4)}}\omega(J,\theta - l_1, Z_1)\frac{\exp(-cl_1)}{|Z - Z_1|} \\ &\times \left(\sqrt{\frac{cl_1 - |Z - Z_1|}{cl_1 + |Z - Z_1|}} + \sqrt{\frac{cl_1 + |Z - Z_1|}{cl_1 - |Z - Z_1|}}\right)K_1\left(\bar{h}_r\frac{(cl_1)^2 - |Z - Z_1|^2}{4}\right)\end{aligned} \tag{287}$$

that becomes in first approximation:

$$|\Psi(\theta - l_1, Z_1)|^2 \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} \simeq \sum_r \frac{\exp\left(-cl_1 - \alpha_r\left((cl_1)^2 - |Z - Z_1|^2\right)\right)}{B_r} H(cl_1 - |Z - Z_1|)\,\omega(J,\theta - l_1, Z_1)$$

where the B_r are constant coefficients and $\alpha_r = \frac{\bar{h}_r}{4}$. As for (276), this also writes:

$$|\Psi(\theta - l_1, Z_1)|^2 \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} \simeq \sum_r \frac{\exp\left(-cl_1 - \alpha_r\left((cl_1)^2 - |Z - Z_1|^2\right)\right)}{D_r} H(cl_1 - |Z - Z_1|)\,\omega(J,\theta - l_1, Z_1) \tag{288}$$

for some constants D_r.

5.2.4 Application: Gaussian transfer function

We apply the previous method to the case of a Gaussian transfer function. The results will be similar to the exponential case, confirming that the results obtained in appendix 5.2.1 are quite general and can be used generally in first approximation

5.2.4.1 Estimation of the poles We can refine (287) by computing more precisely the poles in (283).

To do so, we perform the change of variable:

$$\begin{aligned}u &= \lambda + \lambda_1 \\ v &= \lambda - \lambda_1\end{aligned}$$

before computing the poles, and equation (283) becomes:

$$\begin{aligned}|\Psi(\theta - l_1, Z_1)|^2 \frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2} &= -\frac{A_1}{2iA|Z - Z_1|}\int \omega(J,\theta - l_1, Z_1) \times \frac{\pi(T'(u) - T'(v))}{1 + 2\pi\frac{(T'(u) - T'(v))}{u - v}} \\ &\times \exp\left(i\frac{u}{2}(cl_1 + |Z - Z_1|) + i\frac{v}{2}(cl_1 - |Z - Z_1|)\right)dudv\end{aligned} \tag{289}$$

We first estimate the v integral using the residues theorem. The poles are solutions of:

$$u-v+2\pi\left(T'(u)-T'(v)\right)=0$$

we write the solutions $v_k(u)$ with $k\geqslant 1$. As a consequence (289) rewrites:

$$\begin{aligned}|\Psi(\theta-l_1,Z_1)|^2\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} &= -\sum_{k\geqslant 1}\frac{A_1 i\pi}{2iA|Z-Z_1|}\int\omega(J,\theta-l_1,Z_1)\times\frac{-2\pi(v_k(u)-u)^2}{1+2\pi T''(v_k(u))}\\ &\quad\times\exp\left(i\frac{u}{2}(cl_1+|Z-Z_1|)+i\frac{v_k(u)}{2}(cl_1-|Z-Z_1|)\right)du\end{aligned}$$

To compute (289), we study its two components independently:

$$\begin{aligned}&-\frac{A_1}{iA|Z-Z_1|}\int\omega(J,\theta-l_1,Z_1)\times\frac{\pi T'(u)}{1+2\pi\frac{(T'(u)-T'(v))}{u-v}}\\&\times\exp\left(i\frac{u}{2}(cl_1+|Z-Z_1|)+i\frac{v}{2}(cl_1-|Z-Z_1|)\right)dudv\end{aligned}\tag{290}$$

and:

$$\begin{aligned}&\frac{A_1}{iA|Z-Z_1|}\int\omega(J,\theta-l_1,Z_1)\times\frac{\pi T'(v)}{1+2\pi\frac{(T'(u)-T'(v))}{u-v}}\\&\times\exp\left(i\frac{u}{2}(cl_1+|Z-Z_1|)+i\frac{v}{2}(cl_1-|Z-Z_1|)\right)dudv\end{aligned}\tag{291}$$

In the integral (290), we first estimate the v integral using the residues theorem. The poles are solutions of:

$$1+2\pi\frac{T'(u)-T'(v)}{u-v}=0$$

That is:

$$v+2\pi T'(v)=u+2\pi T'(u)\tag{292}$$

with $v\neq u$.

Now, we consider the following gaussian form for the transfer functions:

$$T(\lambda)=A\exp\left(-\eta\frac{\lambda^2}{4}\right)\left(1-\operatorname{erf}(i\sqrt{\eta}\lambda)\right)\tag{293}$$

and its derivative satisfies:

$$T'(\lambda)=-\eta\frac{\lambda}{2}T(\lambda)-i\sqrt{\eta}$$

As a consequence of these two identities, the solutions of (292) are given by:

$$v\left(1-\pi\eta T(v)\right)=z\tag{294}$$

with:

$$z=u\left(1-\pi\eta T(u)\right)$$

To solve (294) it will be useful to expand $T(\lambda)$ as a series expansion. In first approximation, one has (see Abramovitz stegun):

$$\operatorname{Im}\operatorname{erf}(i\sqrt{\eta}\lambda)\simeq\frac{2}{\pi}\sum_{k=1}^{+\infty}\frac{\exp(-k^2)\sinh k\sqrt{\eta}\lambda}{k}\simeq\sqrt{\frac{\eta}{\pi}}\lambda$$

and:

$$\operatorname{Im}T(\lambda)\simeq A\sqrt{\eta}\left(\frac{1}{2\sqrt{\pi}}\exp\left(-\eta\frac{\lambda^2}{4}\right)\eta\lambda^2\right)>0$$

as a consequence $\operatorname{Im}z>0$ and asymptotically, equation (294) reduces to:

$$v\left(\pi\eta T(v)\right)=-z$$

that is:

$$\left(A\pi\eta v\right)^2 \exp\left(-\eta\frac{v^2}{2}\right)\left(1-\operatorname{erf}\left(i\sqrt{\eta}v\right)\right)^2 = z^2$$

for $\eta << 1$

$$\left(\frac{A\pi\eta v}{2}\right)^2 \exp\left(-\eta\frac{v^2}{2}\right) = z^2$$

and the poles arising in (290) are given by the Whittaker functions W_k:

$$v = \sqrt{-\frac{2}{\eta}W_k\left(\frac{-2z^2}{\left(A\pi\right)^2\eta}\right)}$$

for $k>0$. They are approximatively equal to:

$$v \simeq \pm i\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)}$$

The terms involved in (290) can thus be evaluated at the poles. First, for $\left(A\pi\right)^2 << 1$:

$$\begin{aligned}
\pi\eta\left|T\left(v\right)\right| &\simeq A\pi\eta\left|\exp\left(-\eta\frac{v^2}{4}\right)\right| \\
&\simeq A\pi\eta\exp\left(\ln\left(\frac{\sqrt{2u^2}}{\left(A\pi\right)\sqrt{\eta}}\right)\right) = \sqrt{2\eta}u
\end{aligned}$$

Asymptotically, for $\sqrt{2\eta}u >> 1$, this formula justifies our previous approximation $v\left(1-\pi\eta T\left(v\right)\right) \simeq -v\pi\eta T\left(v\right)$. For $\sqrt{2\eta}u << 1$, the solution is $v=u$ and there is no pole. Second, we have:

$$\begin{aligned}
&\left(1+2\pi\frac{\left(T'\left(u\right)-T'\left(v\right)\right)}{u-v}\right)' \\
&= -2\pi\frac{T''\left(v\right)}{u-v}+2\pi\frac{\left(T'\left(u\right)-T'\left(v\right)\right)}{\left(u-v\right)^2} \\
&= -\frac{1+2\pi T''\left(v\right)}{u-v}
\end{aligned}$$

and (290) becomes:

$$\begin{aligned}
\sum_{k\neq 0} \quad &\frac{2\pi A_1}{A\left|Z-Z_1\right|}\int \omega\left(J,\theta-l_1,Z_1\right)\times\frac{\left(u-v\right)T'\left(u\right)}{2\left(1+T''\left(v\right)\right)} \\
&\times\exp\left(i\frac{u}{2}\left(cl_1+\left|Z-Z_1\right|\right)-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)}\left|cl_1-\left|Z-Z_1\right|\right|\right)du
\end{aligned}$$

Note that for $\left(A\pi\right)^2\eta << 1$, we recover $\delta\left(cl_1-\left|Z-Z_1\right|\right)$ as needed in the lowest order approximation.

The second integral (291) is obtained by inverting the role of u and v. It yields:

$$\begin{aligned}
-\sum_{k\neq 0} \quad &\frac{2\pi A_1}{A\left|Z-Z_1\right|}\int \omega\left(J,\theta-l_1,Z_1\right)\times\frac{\left(u-v\right)T'\left(v\right)}{2\left(1+T''\left(u\right)\right)} \\
&\times\exp\left(-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)}\left(cl_1+\left|Z-Z_1\right|\right)+i\frac{v}{2}\left(cl_1-\left|Z-Z_1\right|\right)\right)du
\end{aligned}$$

and this can be neglected, since $cl_1+\left|Z-Z_1\right|>0$ and for $\left(A\pi\right)^2\eta << 1$ this becomes $\delta\left(cl_1+\left|Z-Z_1\right|\right)=0$.

Gathering the results for (290) and (291), we are left with:

$$
\begin{aligned}
\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2 & \frac{\delta\omega^{-1}\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2} \\
=\sum_{k\neq 0} & \frac{2\pi A_1}{A\left|Z-Z_1\right|}\int\omega\left(J,\theta-l_1,Z_1\right)\times\frac{\left(u-v\right)T'\left(u\right)}{2\left(1+T''\left(v\right)\right)} \\
& \times\exp\left(i\frac{u}{2}\left(cl_1+\left|Z-Z_1\right|\right)-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)}\left|cl_1-\left|Z-Z_1\right|\right|\right)du
\end{aligned}
\tag{295}
$$

Ulitmately, some simplifications can be performed on (295). Actually, we have the following identities for T:

$$
T''\left(\lambda\right)=-\frac{\nu}{2}T\left(\lambda\right)+\left(\nu\frac{\lambda}{2}\right)^2T\left(\lambda\right)+Ai\left(\sqrt{\nu}\right)^3\frac{\lambda}{2}
$$

$$
\begin{aligned}
v\left(1-\pi\eta T\left(v\right)\right) &= u\left(1-\pi\eta T\left(u\right)\right)\simeq u \\
\pi\eta T\left(v\right) &\simeq v-u
\end{aligned}
$$

and this two equations imply that, for $A\left(\sqrt{\eta}\right)^3<<1$:

$$
\begin{aligned}
& 1+2\pi T''\left(v\right) \\
= & 1-\pi\eta T\left(v\right)+2\pi\left(\eta\frac{v}{2}\right)^2T\left(v\right)+2\pi iA\left(\sqrt{\eta}\right)^3\frac{v}{2} \\
\simeq & \left(v-u\right)\left(-1+\pi\eta v\right)\simeq i\left(v-u\right)\sqrt{\frac{2}{\eta}}C
\end{aligned}
\tag{296}
$$

where $C=\sqrt{\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)}$. A consequence of (296) is that:

$$
-\frac{1+2\pi T''\left(v\right)}{u-v}\simeq\frac{1}{i\pi\sqrt{2\eta}C}
$$

Moreover, for $\left(A\pi\right)^2\eta<<1$, the function $T'\left(u\right)$ can be replaced by the multiplication by $i\frac{cl_1+\left|Z-Z_1\right|}{2}$. We are thus led to rewrite (295):

$$
\begin{aligned}
& \left|\Psi\left(\theta-l_1,Z_1\right)\right|^2\frac{\delta\omega^{-1}\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2} \\
= & \frac{A_1}{A}\sum_{k\neq 0}\frac{\left(cl_1+\left|Z-Z_1\right|\right)}{2\left|Z-Z_1\right|}\int\omega\left(J,\theta-l_1,Z_1\right)\times\frac{T\left(u\right)}{\sqrt{2\eta}C} \\
& \times\exp\left(i\frac{u}{2}\left(cl_1+\left|Z-Z_1\right|\right)-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)}\left|cl_1-\left|Z-Z_1\right|\right|\right)du \\
\equiv & \frac{A_1}{A}\Xi\left(\left|Z_1-Z\right|,l_1,\bar{\omega}\right)\omega\left(J,\theta-l_1,Z_1\right)
\end{aligned}
\tag{297}
$$

Remark that, for $\left(A\pi\right)^2\eta<<1$:

$$
\begin{aligned}
& \sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)} \\
& \times\exp\left(-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{\left(A\pi\right)^2\eta}\right)+i\left(2k+1\right)\pi\right)}\left|cl_1-\left|Z-Z_1\right|\right|\right) \\
\simeq & \delta\left(cl_1-\left|Z-Z_1\right|\right)
\end{aligned}
$$

so that one recovers the first order term.

For $(A\pi)^2\eta << 1$, $\Xi(|Z_1 - Z|, l_1, \bar{\omega})$ is a function of $|Z_1 - Z|$ written $\Xi(|Z_1 - Z|, \bar{\omega})$.

Finally, the sum in (297) can be estimated in the following way:

$$\begin{aligned}
&\sum_{k\neq 0}\exp\left(-\sqrt{\frac{2}{\eta}\left(\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)+i(2k+1)\pi\right)}\left|cl_1-|Z-Z_1|\right|\right)\\
=&\sum_{k\neq 0}\exp\left(-\sqrt{\frac{2}{\eta}\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)}\sqrt{1+i\frac{(2k+1)}{\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)}\pi}\left|cl_1-|Z-Z_1|\right|\right)\\
\simeq&\frac{C^2}{\pi}\operatorname{Re}\int_0^1\exp\left(-C\sqrt{\frac{2}{\eta}}\sqrt{1+ix}\left|cl_1-|Z-Z_1|\right|\right)dx\\
=&\frac{C^2}{\pi}\operatorname{Re}\int_0^1\exp\left(-C\sqrt{\frac{2}{\eta}}\left(1+x^2\right)^{\frac{1}{4}}\exp\left(\frac{i}{2}\arctan(x)\right)\left|cl_1-|Z-Z_1|\right|\right)dx
\end{aligned}$$

with:

$$C=\sqrt{\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)}$$

The upper bound of the integral is set to 1, in agreement with our approximation $\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right) >> 1$. It amounts to neglect the poles for $k >> 1$, whose contributions are decreasing quicly with k as given by oscillatory integrals of frequencies proportional to k.

By a change of variable, the last integral is also given by:

$$\frac{2C^2}{\pi}\operatorname{Re}\int_0^1\exp\left(-C\sqrt{\frac{2}{\eta}}\left(\sqrt{1+v^2}+iv\right)\left|cl_1-|Z-Z_1|\right|\right)\left(\sqrt{1+v^2}+\frac{v^2}{\sqrt{1+v^2}}\right)dv$$

and we are left with the estimation for the first vertex:

$$\begin{aligned}
&|\Psi(\theta-l_1,Z_1)|^2\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2} \qquad (298)\\
=&\frac{A_1}{A}\sum_{k\neq 0}\frac{(cl_1+|Z-Z_1|)}{2|Z-Z_1|}\int\omega(J,\theta-l_1,Z_1)\times\frac{T(u)}{\sqrt{2\eta}C}du\\
&\frac{2C^2}{\pi}\operatorname{Re}\int_0^1\exp\left(-C\sqrt{\frac{2}{\eta}}\left(\sqrt{1+v^2}+iv\right)\left|cl_1-|Z-Z_1|\right|\right)\left(\sqrt{1+v^2}+\frac{v^2}{\sqrt{1+v^2}}\right)dv
\end{aligned}$$

5.2.4.2 Gaussian approximation We can estimate the integral $\int_0^1 dv$ in (298) by integrating between 0 and $+\infty$.

$$\begin{aligned}
&\frac{2C^2}{\pi}\operatorname{Re}\int_0^{+\infty}\exp\left(-C\sqrt{\frac{2}{\eta}}\left(\sqrt{1+v^2}+iv\right)\left|cl_1-|Z-Z_1|\right|\right)\left(2\sqrt{1+v^2}-\frac{1}{\sqrt{1+v^2}}\right)dv\\
=&\frac{2C^2}{\pi}\operatorname{Re}\int_0^{+\infty}\exp\left(-C\sqrt{\frac{2}{\eta}}\left(\sqrt{1+v^2}a+ivb\right)\right)\left(2\sqrt{1+v^2}-\frac{1}{\sqrt{1+v^2}}\right)dv
\end{aligned}$$

with $a = b = |cl_1 - |Z - Z_1||$. The last integral can be rewritten:

$$\begin{aligned}
& \left(-\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}}\int da\right)\frac{2C^2}{\pi}\operatorname{Re}\int_0^{+\infty}\exp\left(-C\sqrt{\frac{2}{\eta}}\left(\sqrt{1+v^2}a + ivb\right)\right)dv \\
= & \left(-\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}}\int da\right)\frac{2C^2}{\pi b}\operatorname{Re}\int_0^{+\infty}\exp\left(-C\sqrt{\frac{2}{\eta}}\left(\sqrt{b^2+v^2}\frac{a}{b} + iv\right)\right)dv \\
\simeq & \left(-\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}}\int da\right)\frac{2C^2}{\pi b}\operatorname{Re}\int_0^{+\infty}\exp\left(-C\sqrt{\frac{2}{\eta}}\left(-\left(\left(\frac{a}{b}+i\right)v + \frac{ab}{2v}\right)\right)\right)dv
\end{aligned} \tag{299}$$

for $a \simeq b << 1$. We use that:

$$\begin{aligned}
& \left(-\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}}\int da\right)\frac{2C^2}{\pi}\operatorname{Re}\int_0^{+\infty}\exp\left(-C\sqrt{\frac{2}{\eta}}\left(-\left((a+ib)v + \frac{a}{2v}\right)\right)\right)dv \\
= & \left(-\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}} + C\sqrt{\frac{2}{\eta}}\int da\right)\frac{2C^2}{\pi}\operatorname{Re}\sqrt{\frac{2a}{a+ib}}K_1\left(2C\sqrt{\frac{a(a+ib)}{\eta}}\right)
\end{aligned} \tag{300}$$

where K_1 is a modified Bessel function, and that the following identity holds for K_1:

$$\sqrt{\frac{2a}{a+ib}}K_1\left(2C\sqrt{\frac{a(a+ib)}{\eta}}\right) \simeq \sqrt{\frac{2a}{a+ib}}\sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}}\exp\left(-2C\sqrt{\frac{a(a+ib)}{\eta}}\right)$$

for $C >> 1$. Then computing the integral $\int da$ in (300) yields:

$$\begin{aligned}
& C\sqrt{\frac{2}{\eta}}\int da\left(\frac{\sqrt{\frac{2a}{a+ib}}\sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}}}{-\frac{C}{\eta}\frac{2a+ib}{\sqrt{\frac{a}{\eta}(a+ib)}}}\left(-\frac{C}{\eta}\frac{2a+ib}{\sqrt{\frac{a}{\eta}(a+ib)}}\right)\exp\left(-2C\sqrt{\frac{a(a+ib)}{\eta}}\right)\right) \\
\simeq & C\sqrt{\frac{2}{\eta}}\frac{\sqrt{\frac{2a}{a+ib}}\sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}}}{-\frac{C}{\eta}\frac{2a+ib}{\sqrt{\frac{a}{\eta}(a+ib)}}}\exp\left(-2C\sqrt{\frac{a(a+ib)}{\eta}}\right) \\
= & -2\sqrt{\pi}a\frac{\sqrt{\frac{1}{4C\sqrt{\frac{a}{\eta}(a+ib)}}}}{2a+ib}\exp\left(-2C\sqrt{\frac{a(a+ib)}{\eta}}\right)
\end{aligned}$$

for $C >> 1$. For $a = b$, this identity reduces to:

$$-\sqrt{\pi}\frac{\frac{2}{5} - \frac{1}{5}i}{\sqrt[4]{(1+i)}}\sqrt{\frac{\sqrt{\eta}}{Ca}}\exp\left(-2\frac{Ca}{\sqrt{\eta}}\sqrt{1+i}\right) \tag{301}$$

The derivative arising in (300) can be estimated by:

$$
\begin{aligned}
&-\frac{2\partial_a}{C\sqrt{\frac{2}{\eta}}}\left(\sqrt{\frac{2a}{a+ib}}\sqrt{\frac{\pi}{4C\sqrt{\frac{a(a+ib)}{\eta}}}}\exp\left(-2C\sqrt{\frac{a\left(a+ib\right)}{\eta}}\right)\right)\\
=\ &-\frac{1}{8}\sqrt{\left(\frac{1}{2}-\frac{1}{2}i\right)}\frac{\sqrt{\pi}}{C^2a^3\eta}\frac{\exp\left(-2\sqrt{(1+i)}C\sqrt{\frac{a^2}{\eta}}\right)}{\left(\frac{1}{\eta}\right)^{\frac{3}{2}}\sqrt{\frac{1}{C\sqrt{\frac{a^2}{\eta}}}}}\left((1+2i)\left((1+i)\right)^{\frac{3}{4}}\eta\sqrt{\frac{a^2}{\eta}}-(12-4i)\sqrt[4]{(1+i)}Ca^2\right)\\
\simeq\ &\frac{1}{8}\sqrt{\left(\frac{1}{2}-\frac{1}{2}i\right)}\frac{\sqrt{\pi}}{C^2a^3\eta}\frac{\exp\left(-2\sqrt{(1+i)}C\sqrt{\frac{a^2}{\eta}}\right)}{\left(\frac{1}{\eta}\right)^{\frac{3}{2}}\sqrt{\frac{1}{C\sqrt{\frac{a^2}{\eta}}}}}\left((12-4i)\sqrt[4]{(1+i)}Ca^2\right)\\
=\ &\frac{\sqrt{\pi}}{8}\sqrt{\left(\frac{1}{2}-\frac{1}{2}i\right)}\frac{\exp\left(-2\sqrt{\frac{(1+i)}{\eta}}Ca\right)}{\sqrt{\frac{Ca}{\sqrt{\eta}}}}\left((12-4i)\sqrt[4]{(1+i)}\right) \qquad (302)
\end{aligned}
$$

Gathering (301) and (302), we find that for $C >> 1$, For $a=b=|cl_1-|Z-Z_1||$, we find for (300):

$$
\begin{aligned}
&C^2\frac{\sqrt{130}}{5\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{4}}\frac{\exp\left(-\frac{2^{\frac{9}{8}}\cos\left(\frac{\pi}{8}\right)C}{\sqrt{\eta}}\left|cl_1-|Z-Z_1|\right|\right)}{\sqrt{\frac{C|cl_1-|Z-Z_1||}{\sqrt{\eta}}}}\cos\left(\frac{2^{\frac{9}{8}}\cos\left(\frac{\pi}{8}\right)C}{\sqrt{\eta}}\left|cl_1-|Z-Z_1|\right|\right)\\
=\ &C^2\frac{\sqrt{65\sqrt{2}}}{5\sqrt{\pi}}\frac{\exp\left(-\frac{2^{\frac{3}{8}}\sqrt{\sqrt{2}+1}C}{\sqrt{\eta}}\left|cl_1-|Z-Z_1|\right|\right)}{\sqrt{\frac{C|cl_1-|Z-Z_1||}{\sqrt{\eta}}}}\cos\left(\frac{2^{\frac{3}{8}}\sqrt{\sqrt{2}+1C}}{\sqrt{\eta}}\left|cl_1-|Z-Z_1|\right|\right)
\end{aligned}
$$

In the sequel, for $(A\pi)^2\eta << 1$, we approximate:

$$C=\sqrt{\ln\left(\frac{2u^2}{(A\pi)^2\eta}\right)}\simeq\sqrt{\ln\left(\frac{2}{(A\pi)^2\eta}\right)}$$

Finally, the integral over u in (298) is:

$$\int\frac{T(u)}{\sqrt{2\eta}C}\exp\left(i\frac{u}{2}\left(cl_1+|Z-Z_1|\right)\right)du=\frac{1}{\sqrt{2\eta}C}\hat{T}\left(\frac{cl_1+|Z-Z_1|}{2}\right)du$$

so that, using that

$$\hat{T}_1\left(\frac{cl_1+|Z-Z_1|}{2}\right)=\frac{A_1}{A}\hat{T}\left(\frac{cl_1+|Z-Z_1|}{2}\right)$$

The result for (298) is:

$$
\begin{aligned}
&\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2}\simeq\frac{\sqrt{65}}{5\sqrt{\pi}2^{\frac{3}{8}}\left(\sqrt{2}+1\right)^{\frac{1}{4}}}D\frac{\exp\left(-D\left|cl_1-|Z-Z_1|\right|\right)}{\sqrt{\frac{C|cl_1-|Z-Z_1||}{\sqrt{\eta}}}}\cos\left(D\left|cl_1-|Z-Z_1|\right|\right)\\
&\times\frac{\left(cl_1+|Z-Z_1|\right)}{2\left|Z-Z_1\right|}\hat{T}_1\left(\frac{cl_1+|Z-Z_1|}{2}\right)
\end{aligned}
$$

where:

$$D=\frac{2^{\frac{3}{8}}\sqrt{\sqrt{2}+1C}}{\sqrt{\eta}}$$

We also write this result in a more compact form:

$$\frac{\delta\omega^{-1}(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2}=\frac{A_1}{A}\Xi\left(\left|Z_1-Z\right|,l_1,\bar{\omega}\right)\omega\left(J,\theta-l_1,Z_1\right) \tag{303}$$

for $\frac{|Z_1-Z|}{cl_1}<1$, and 0 otherwise, with:

$$\begin{aligned}\Xi\left(\left|Z_1-Z\right|,l_1,\bar{\omega}\right) &= \frac{\sqrt{65}}{5\sqrt{\pi}2^{\frac{3}{8}}\left(\sqrt{2}+1\right)^{\frac{1}{4}}}\frac{D\exp\left(-D\left|cl_1-\left|Z-Z_1\right|\right|\right)}{\sqrt{D\left|cl_1-\left|Z-Z_1\right|\right|}}\cos\left(D\left|cl_1-\left|Z-Z_1\right|\right|\right) \\ &\times\frac{\left(cl_1+\left|Z-Z_1\right|\right)}{2\left|Z-Z_1\right|}\hat{T}\left(\frac{cl_1+\left|Z-Z_1\right|}{2}\right)\end{aligned} \tag{304}$$

Similarly:

$$\frac{\delta\omega(J,\theta,Z)}{\delta\left|\Psi(\theta-l_1,Z_1)\right|^2}=\Xi\left(\left|Z_1-Z\right|,l_1,\bar{\omega}\right)\omega\left(J,\theta-l_1,Z_1\right) \tag{305}$$

The appearance of the $\cos\left(D\left|cl_1-\left|Z-Z_1\right|\right|\right)$ in (304) is a consequence of our approximation computing the integral between 0 and $+\infty$. This approximation breaks down when the cos function becomes negative. As a consequence, for $D\left|cl_1-\left|Z-Z_1\right|\right|>\frac{\pi}{2}$, we can set $\Xi\left(\left|Z_1-Z\right|,l_1,\bar{\omega}\right)\simeq 0$.

As stated in the beginning of this paragraph, formula (304) is similar to the case of an exponential transfer function.

Appendix 6 Non local expansion for $\omega(\theta,Z)$

6.1 n-th derivatives of at $|\Psi|^2=0$

6.1.1 General formula

Based on the results of Appendix 5, we can now compute $\omega(J,\theta,Z)$, $\omega^{-1}(J,\theta,Z)$ and their derivatives $\frac{\delta^n\omega(J,\theta,Z)}{\prod_{i=1}^n\delta|\Psi(\theta-l_i,Z_i)|^2}$ and $\frac{\delta^n\omega^{-1}(J,\theta,Z)}{\prod_{i=1}^n\delta|\Psi(\theta-l_i,Z_i)|^2}$. It allows to compute the expansion of the effective action, and also to study the solutions of (84) without the locality assumption.

6.1.1.1 Series expansion for the first order derivative of $\omega(\theta,Z)$ Recall that $\omega(\theta,Z)$ is solution of (85):

$$\begin{aligned}\omega(\theta,Z) &= F\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(\theta,Z)}\right. \\ &\left.\times W\left(\frac{\omega(\theta,Z)}{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\bar{\mathcal{G}}_0(0,Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned} \tag{306}$$

$$\bar{\mathcal{G}}_0(0,Z_1)=\mathcal{G}_0(0,Z_1)+X_0$$

and $\omega^{-1}(\theta,Z)$ is solution of:

$$\begin{aligned}\omega^{-1}(\theta,Z) &= G\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(\theta,Z)}\right. \\ &\left.\times W\left(\frac{\omega(\theta,Z)}{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\bar{\mathcal{G}}_0(0,Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

To find the internal dynamics of the system we will consider $J(\theta) = J$, a constant external current, usually $J = 0$. We use a series expansion in $\left|\Psi\left(\theta^{(j)}, Z_1\right)\right|^2$ of the right hand side of (306) and write:

$$\begin{aligned}\omega\left(\theta^{(i)}, Z\right) &= \omega\left(\theta^{(i)}, Z\right)_{|\Psi|^2=0} \\ &+ \int \sum_{n=1}^{\infty} \left(\frac{\delta^n \omega(J, \theta, Z)}{\prod_{i=1}^{n} \delta \left|\Psi(\theta - l_i, Z_i)\right|^2} \right)_{|\Psi|^2=0} \prod_{i=1}^{n} \left|\Psi(\theta - l_i, Z_i)\right|^2 \end{aligned} \tag{307}$$

The first term (307), i.e. $\omega\left(\theta^{(i)}, Z\right)_{|\Psi|^2=0}$, is a solution of:

$$F\left(J + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)} W\left(\frac{\omega(\theta, Z)}{\omega\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)}\right) \left(\bar{\mathcal{G}}_0(0, Z_1)\right) dZ_1\right)$$

One solution is the static frequency (88) solution of:

$$\begin{aligned}\omega(J, Z) &= F\left(J + \frac{\kappa}{N} \int T(Z, Z_1) \frac{\omega(Z_1)}{\omega(Z)} W\left(\frac{\omega(Z)}{\omega(Z_1)}\right) \bar{\mathcal{G}}_0(0, Z_i) dZ_1\right) \\ &\equiv F[J, \omega, Z]\end{aligned}$$

but any time dependent solution for $|\Psi|^2 = 0$ is also possible. This arises for non constant current $J(\theta)$. Equation (307) is the expansion of $\omega\left(\theta^{(i)}, Z\right)$ around this solution, the dynamics depending on $\left|\Psi\left(\theta^{(j)}, Z_1\right)\right|^2$. We set:

$$\omega\left(\theta^{(i)}, Z\right)_{|\Psi|^2=0} = \omega_0(J, Z)$$

The first derivative $\frac{\delta \omega(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2}$ in (307) has been computed in Appendix 5. It is given by:

$$\begin{aligned}\frac{\delta \omega(J, \theta, Z)}{\delta \left|\Psi(\theta - l_1, Z_1)\right|^2} &= \sum_{n=1}^{\infty} \int \frac{\omega\left(J, \theta - \sum_{l=1}^{n} \frac{\left|Z^{(l-1)} - Z^{(l)}\right|}{c}, Z_1\right)}{\left(\bar{\mathcal{G}}_0(0, Z_1) + \left|\Psi(\theta - l_1, Z_1)\right|^2\right)} \\ &\times \prod_{l=1}^{n} \hat{T}\left(\theta - \sum_{j=1}^{l-1} \frac{\left|Z^{(j-1)} - Z^{(j)}\right|}{c}, Z^{(l-1)}, Z^{(l)}, \omega, \Psi\right) \delta\left(l_1 - \sum_{l=1}^{n} \frac{\left|Z^{(l-1)} - Z^{(l)}\right|}{c}\right) \prod_{l=1}^{n-1} dZ^{(l)}\end{aligned} \tag{308}$$

where:

$$\begin{aligned}&\hat{T}(\theta, Z, Z_1 \omega, \Psi) \\ &= \frac{\frac{\kappa}{N} \omega(J, \theta, Z) T(Z, Z_1) \bar{W}'\left(\frac{\omega\left(J, \theta - \frac{|Z - Z_1|}{c}, Z_1\right)}{\omega(J, \theta, Z)}\right) F'[J, \omega, \theta, Z, \Psi] \left(\bar{\mathcal{G}}_0(0, Z_1) + \left|\Psi\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)\right|^2\right)}{\omega^2(J, \theta, Z) + \left(\int \frac{\kappa}{N} \omega\left(J, \theta - \frac{|Z - Z'|}{c}, Z'\right) \bar{W}'\left(\frac{\omega\left(J, \theta - \frac{|Z - Z'|}{c}, Z'\right)}{\omega(J, \theta, Z)}\right) \left(\bar{\mathcal{G}}_0(0, Z') + \left|\Psi\left(\theta - \frac{|Z - Z'|}{c}, Z'\right)\right|^2\right) T(Z, Z') dZ'\right)}\end{aligned}$$

with the convention that $Z^{(0)} = Z$ and $Z^{(n)} = Z_1$. The derivative (308) was then evaluated in Appendix 5 using combinations of K_1 functions, but for the purpose of the computation of the successive derivatives of $\omega(J, \theta, Z)$, we will work, temporarily, with the general formula (308). Equation (308) yield recursively $\frac{\delta \omega(J, \theta, Z)}{\delta|\Psi(\theta - l_1, Z_1)|^2}$ in terms of past frequencies. Applied to the case $|\Psi|^2 = 0$, the factor (309) simplifies:

$$\begin{aligned}\hat{T}(\theta, Z, Z_1, \omega_0) &\equiv \hat{T}(\theta, Z, Z_1 \omega_0, 0) \\ &= \frac{\frac{\kappa}{N} \omega_0(J, \theta, Z) \bar{W}'\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) T(Z, Z_1) F'[J, \omega, \theta, Z, \Psi] \bar{\mathcal{G}}_0(0, Z_1)}{\omega_0^2(J, \theta, Z) + \left(\int \frac{\kappa}{N} \omega_0(J, Z') \bar{W}'\left(\frac{\omega_0(Z)}{\omega_0(Z_1)}\right) \left(\bar{\mathcal{G}}_0(0, Z')\right) T(Z, Z') dZ'\right) F'[J, \omega, \theta, Z, \Psi]}\end{aligned} \tag{310}$$

or in first aproximation:

$$\begin{aligned}\hat{T}\left(\theta, Z, Z_1, \omega_0, \Psi\right) &\equiv \hat{T}\left(Z, Z_1, \omega_0\right) \\ &\simeq \frac{\frac{\kappa}{N} T\left(Z, Z_1\right) F'\left[J, \omega_0, \theta, Z\right] \bar{\mathcal{G}}_0\left(0, Z_1\right)}{\omega_0\left(J, Z\right)}\end{aligned} \tag{311}$$

and (308) becomes:

$$\begin{aligned}\left(\frac{\delta\omega\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2}\right)_{|\Psi|^2=0} &= \sum_{n=1}^{\infty}\int\frac{\omega_0\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)}{\bar{\mathcal{G}}_0\left(0,Z_1\right)} \\ &\times\prod_{l=1}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0,0\right) \\ &\times\delta\left(l_1-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}\end{aligned} \tag{312}$$

6.1.1.2 Graphical representation of the successive derivatives The n-th term in (312) can be understood graphically as a sum over the set of broken paths with n segments, each path linking $Z^{(l-1)}$ and $Z^{(l)}$ during a timespan of $\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}$. To each point of the segment, we associate the factor:

$$\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0,\Psi\right)\simeq\frac{\frac{\kappa}{N}T\left(Z^{(l-1)},Z^{(l)}\right)F'\left[J,\omega_0,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)}\right]\bar{\mathcal{G}}_0\left(0,Z^{(l)}\right)}{\omega_0\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)}\right)} \tag{313}$$

Ultimately, the product of factor is multiplied by the frequency at the last point:

$$\frac{\omega_0\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)}{\bar{\mathcal{G}}_0\left(0,Z_1\right)} \tag{314}$$

and by $\left|\Psi\left(\theta-l_1,Z_1\right)\right|^2$. The integrals over the points $Z^{(l)}$ and the sum over n, the length of the broken paths, yield the first order contribution to the expansion (307).

The next terms in the expansion of (307) are the derivatives $\left(\frac{\delta^n\omega(J,\theta,Z)}{\prod_{i=1}^{n}\delta|\Psi(\theta-l_i,Z_i)|^2}\right)_{|\Psi|^2=0}$ which are obtained by successive derivations of (308) and (309) by $\left|\Psi\left(\theta-l_2,Z_2\right)\right|^2$ and evaluated at $\left|\Psi\right|^2=0$. The l_i are ordered such that $l_1<...<l_n$. These derivatives are obtained by differentiating either:

$$\omega\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_n\right)$$

or the successive factors:

$$\prod_{l=1}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)$$

The first possibility amounts to write $\frac{\delta\omega(J,\theta-l_1,Z_1)}{\delta|\Psi(\theta-l_2,Z_2)|^2}$ using (308). Graphically it amounts to write broken lines from Z_1 to Z_2 and associate to each broken line the factor (313), (314) and $\left|\Psi\left(\theta-l_2,Z_2\right)\right|^2$.

The second possibility is obtained by computing for each l:

$$\frac{\delta\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)}{\delta\left|\Psi\left(\theta-l_{2},Z_{2}\right)\right|^{2}} \tag{315}$$

Which can be written as:

$$\begin{aligned}&\frac{\delta\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)}{\delta\left|\Psi\left(\theta-l_{2},Z_{2}\right)\right|^{2}}\\ =&\int d\Delta dZ'\frac{\delta\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)}{\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}-\Delta,Z'\right)}\frac{\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}-\Delta,Z'\right)}{\delta\left|\Psi\left(\theta-l_{2},Z_{2}\right)\right|^{2}}\end{aligned}$$

This derivative can be described graphically by assigning to some point $Z^{(l)}$ of the initial line the factor:

$$\frac{\delta\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)}{\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}-\Delta,Z'\right)}$$

issuing a new succession of segments representing $\frac{\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}-\Delta,Z'\right)}{\delta\left|\Psi\left(\theta-l_{2},Z_{2}\right)\right|^{2}}$ and then summing over Δ and Z'. In first approximation, we can set $\Delta=0$ and Z', so that the factor is:

$$\left(\frac{\delta\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)}{\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)}\right)}\right)_{|\Psi|^{2}=0}$$

and the new succession of segments represents $\frac{\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)}\right)}{\delta\left|\Psi\left(\theta-l_{2},Z_{2}\right)\right|^{2}}$.

More generally, differentiating successively $\hat{T}\left(\theta,Z,Z_{1}\omega,|\Psi|^{2}\right)$, corresponds to insert the vertices:

$$\frac{\delta^{k}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega,\Psi\right)}{\prod_{i=1}^{k}\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}-\Delta_{l},Z_{l}\right)}\simeq\frac{\delta^{k}\left(\frac{\frac{\kappa}{N}T\left(Z,Z^{(l)}\right)F'\left[J,\theta,\omega_{0},Z^{(l)}\right]\bar{\mathcal{G}}_{0}\left(0,Z^{(l)}\right)}{\omega_{0}\left(J,\theta,Z^{(l)}\right)}\right)}{\delta^{k}\omega_{0}\left(J,\theta,Z^{(l)}\right)}$$

with k new segments representing $\frac{\delta\omega\left(J,\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}-\Delta_{l},Z_{l}\right)}{\delta\left|\Psi\left(\theta-l_{l},Z_{l}\right)\right|^{2}}$.

Gathering the two possibilities forementionned and iterating this procedures yields a graphical representation for:

$$\left(\frac{\delta^{n}\omega\left(J,\theta,Z\right)}{\prod_{i=1}^{n}\delta\left|\Psi\left(\theta-l_{i},Z_{i}\right)\right|^{2}}\right)_{|\Psi|^{2}=0}\prod_{i=1}^{n}\left|\Psi\left(\theta-l_{i},Z_{i}\right)\right|^{2} \tag{316}$$

We associate the squared field $\left|\Psi\left(\theta-l_{i},Z_{i}\right)\right|^{2}$ to each point Z_{i} . For $m=1,...,n$, we draw m lines. At least one of them is starting from Z. These lines are composed of an arbitrary number of segments and all the points Z_i are crossed by one line. Each line ends at a point Z_i. The starting points of the lines have to branch either at Z, either at some point of an other line. There are m branching points of valence k including the starting point at Z Apart from Z the branching points have valence $3,...,n-1$. To each line i of length L_i, we associate the factor:

$$\begin{aligned}
F\left(line_i\right) &= \prod_{l=1}^{L_i} \frac{\frac{\kappa}{N} T\left(Z^{(l-1)}, Z^{(l)}\right) F'\left[J, \omega_0, \theta - \sum_{j=1}^{l-1} \frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}, Z^{(l-1)}\right] \bar{\mathcal{G}}_0\left(0, Z^{(l)}\right)}{\omega_0\left(J, \theta - \sum_{j=1}^{l-1} \frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c}, Z^{(l-1)}\right)} \qquad (317) \\
&\quad \times \frac{\omega_0\left(J, \theta - \sum_{l=1}^{L_i} \frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}, Z_i\right)}{\bar{\mathcal{G}}_0\left(0, Z_i\right)} \\
&= \prod_{l=1}^{L_i} \hat{T}\left(\theta - \sum_{j=1}^{l-1} \frac{\left|Z^{(j-1)} - Z^{(j)}\right|}{c}, Z^{(l-1)}, Z^{(l)}, \omega_0, \Psi\right) \frac{\omega_0\left(J, \theta - \sum_{l=1}^{L_i} \frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}, Z_i\right)}{\bar{\mathcal{G}}_0\left(0, Z_i\right)}
\end{aligned}$$

and to each branching point $(X, \theta) = B$ of valence $k+2$ arising in the expansion, we associate the factor:

$$F\left((X, \theta)\right) = \frac{\delta^k \left(\frac{\frac{\kappa}{N} T\left(Z, Z^{(l)}\right) F'\left[J, \theta, \omega_0, Z^{(l)}\right] \bar{\mathcal{G}}_0\left(0, Z^{(l)}\right)}{\omega_0\left(J, \theta, Z^{(l)}\right)}\right)}{\delta^k \omega_0\left(J, \theta, Z^{(l)}\right)} \qquad (318)$$

and (316) writes:

$$\begin{aligned}
&\left(\frac{\delta^n \omega\left(J, \theta, Z\right)}{\prod_{i=1}^{n} \delta \left|\Psi\left(\theta - l_i, Z_i\right)\right|^2}\right)_{|\Psi|^2=0} \prod_{i=1}^{n} \left|\Psi\left(\theta - l_i, Z_i\right)\right|^2 \\
&= \left(\sum_{m=1}^{n} \sum_{i=1}^{m} \sum_{(line_1, \ldots, line_m)} \prod_i F\left(line_i\right) \prod_B F\left(B\right)\right) \prod_{i=1}^{n} \left|\Psi\left(\theta - l_i, Z_i\right)\right|^2 \qquad (319)
\end{aligned}$$

The integral over the branch points is implicit. The factor $F(B)$ for a branch point B is defined in (318) The graphical representation is generic. While integrating over the set of lines, the degenerate case of lines that share some segments is taken into account.

6.1.2 Approximate expression

The results of the section 5 can then be used with (319) to compute:

$$\frac{\delta^n \omega\left(J, \theta, Z\right)}{\prod_{i=1}^{n} \delta \left|\Psi\left(\theta - l_i, Z_i\right)\right|^2}$$

in the approximation of the dominant contribution. To each line from a branching point $\theta - l'_j, Z'_j$ to $\theta - l_i, Z_i$ (the branching point can be one of the $\theta - l_i, Z_i$) we associate a factor of the type, as in (275):

$$\frac{\exp\left(-c\left(l_i - l'_j\right) - \gamma\left(c\left(l_i - l'_j\right) - \left|Z'_j - Z_i\right|\right)\right)}{B} H\left(cl_1 - \left|Z - Z_1\right|\right)$$

The dominant contribution is obtained when the set $\{l'_j, Z'_j\}$ is equal to $\{l_j, Z_j\}$ and the product over the branching points yields a contribution whose form is:

$$\frac{\delta^n \omega(J,\theta,Z)}{\prod_{i=1}^{n} \delta |\Psi(\theta - l_i, Z_i)|^2} \simeq \frac{\exp\left(-cl_n - \gamma\left(\sum_{i=1}^{n-1}\left((c(l_i - l_{i+1}))^2 - |Z_i - Z_{i+1}|^2\right)\right)\right)}{B^n} \tag{320}$$
$$\times H\left(cl_n - \sum_{i=1}^{n-1}|Z_i - Z_{i+1}|\right)\prod_{i=1}^{n}\frac{\omega_0(J,\theta - l_i, Z_i)}{\bar{\mathcal{G}}_0(0,Z_i)}$$

with $Z_1 = Z$ and $l_n > ... > l_1$ and B a constant coefficient (see (276)).

Formula (266) shows that the previous computations are also valid for the derivatives of $\omega^{-1}(J,\theta,Z)$. We thus obtain the generalization of (276):

$$\frac{\delta^n \omega^{-1}(J,\theta,Z)}{\prod_{i=1}^{n} \delta |\Psi(\theta - l_i, Z_i)|^2} \simeq \frac{\exp\left(-cl_n - \alpha\left(\sum_{i=1}^{n-1}\left((c(l_i - l_{i+1}))^2 - |Z_i - Z_{i+1}|^2\right)\right)\right)}{D^n} \tag{321}$$
$$\times H\left(cl_n - \sum_{i=1}^{n-1}|Z_i - Z_{i+1}|\right)\prod_{i=1}^{n}\frac{\omega_0^{-1}(J,\theta - l_i, Z_i)}{\bar{\mathcal{G}}_0(0,Z_i)}$$

The only difference is the appearance of different coefficients α and D in the expression.

6.2 Equation for $\omega(\theta, Z)$

6.2.1 Reordering the graphical sum (319)

We now sum the series expansion (307):

$$\omega\left(\theta^{(i)}, Z\right) = \omega\left(\theta^{(i)}, Z\right)_{|\Psi|^2=0} + \int \sum_{n=1}^{\infty}\left(\frac{\delta^n \omega(J,\theta,Z)}{\prod_{i=1}^{n} \delta |\Psi(\theta - l_i, Z_i)|^2}\right)_{|\Psi|^2=0} \prod_{i=1}^{n}|\Psi(\theta - l_i, Z_i)|^2 \tag{322}$$

by reordering the sums in the RHS of (319).

To do so, we first compute the sum over the lines between (Z,θ) and (Z_1,θ_1) and of given length $L_i = n$ of the product of factors $\hat{T}\left(\theta - \sum_{j=1}^{l-1}\frac{|Z^{(j-1)} - Z^{(j)}|}{c}, Z^{(l-1)}, Z^{(l)}, \omega_0, \Psi\right)$ in $F(line_i)$ (see (317) for the definition of $F(line_i)$). This sum is computed in (312). We call the result $G_0^{(n)}((Z,\theta),(Z_1,\theta_1))$, so that:

$$G_0^{(n)}((Z,\theta),(Z_1,\theta_1)) = \int \prod_{l=1}^{n}\hat{T}\left(\theta - \sum_{j=1}^{l-1}\frac{|Z^{(j-1)} - Z^{(j)}|}{c}, Z^{(l-1)}, Z^{(l)}, \omega_0\right)$$
$$\times\delta\left((\theta - \theta_1) - \sum_{l=1}^{n}\frac{|Z^{(l-1)} - Z^{(l)}|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}$$
$$= \int \prod_{l=1}^{n}\hat{T}\left(\theta - \sum_{j=1}^{l-1}\frac{|Z^{(j-1)} - Z^{(j)}|}{c}, Z^{(l-1)}, Z^{(l)}, \omega_0\right)\delta\left(\left(\theta^{(l)} - \theta^{(l-1)}\right) - \frac{|Z^{(l-1)} - Z^{(l)}|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}d\theta_l$$

with $\left(Z^{(0)}, \theta^{(0)}\right) = (Z,\theta)$ and $\left(Z^{(n)}, \theta^{(n)}\right) = (Z_1,\theta_1)$.

Then, we sum over the length n of the lines and the factor associated to the sum of lines, written $G_0((Z,\theta),(Z_1,\theta_1))$, is:

$$\begin{aligned}G_0((Z,\theta),(Z_1,\theta_1)) &= \sum_{n=1}^{\infty}\int\prod_{l=1}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0\right)\\ &\quad\times\delta\left(\left(\theta^{(l)}-\theta^{(l-1)}\right)-\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}d\theta_l\end{aligned}$$

The function $G_0((Z,\theta),(Z_1,\theta_1))$ is a series expansion that can be summed:

$$G_0((Z,\theta),(Z_1,\theta_1))=\hat{T}\left(1-\hat{T}\right)^{-1}((Z,\theta),(Z_1,\theta_1)) \tag{323}$$

with:

$$\hat{T}\left(\left(Z^{(l-1)},\theta^{(l-1)}\right),\left(Z^{(l)},\theta^{(l)}\right)\right)=\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0\right)\delta\left(\left(\theta^{(l)}-\theta^{(l-1)}\right)-\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)$$

As a consequence, equation (319) can be rewritten as a sum over the branch points.:

$$\begin{aligned}\omega\left(\theta^{(i)},Z\right)-\omega\left(\theta^{(i)},Z\right)_{|\Psi|^2=0} &= \int\sum_n\left(\frac{\delta^n\omega(J,\theta,Z)}{\prod_{i=1}^{n}\delta|\Psi(\theta-l_i,Z_i)|^2}\right)_{|\Psi|^2=0}\prod_{i=1}^{n}|\Psi(\theta-l_i,Z_i)|^2\,dl_idZ_i \qquad (324)\\ &= \left(\sum_{m=1}^{n}\sum_{i=1}^{m}\sum_{B}\sum_{\left(\overline{line_1},\ldots,\overline{line_m}\right)}\prod_i G_0\left(\overline{line_i}\right)\prod_B F(B)\right)\prod_{i=1}^{n}|\Psi(\theta-l_i,Z_i)|^2\end{aligned}$$

The sum $\sum_{\left(\overline{line_1},\ldots,\overline{line_m}\right)}$ is over the finite set of m segments connecting two branch points and respecting the constraint given above (317). If $\overline{line_i}$ connects two branch points $((X_1,\theta_1),(X_2,\theta_2))$, then $G_0\left(\overline{line_i}\right)$ is equal to $G_0((X_1,\theta_1),(X_2,\theta_2))$. At each branch point we insert $\frac{|\Psi(\theta-l_k,Z_k)|^2}{\bar{\mathcal{G}}_0(0,Z_k)}$ and for a terminal point $\frac{\omega_0(J,\theta-l_k,Z_k)|\Psi(\theta-l_k,Z_k)|^2}{\bar{\mathcal{G}}_0(0,Z_k)}$. We will normalize $|\Psi|^2$ by $\bar{\mathcal{G}}_0$, so that $|\Psi(\theta-l_k,Z_k)|^2$ will stand for $\frac{|\Psi(\theta-l_k,Z_k)|^2}{\bar{\mathcal{G}}_0(0,Z_k)}$.

Now the sums in (324) can be reordered in the following way. We consider the lines from (θ,Z) to a final point, and sum over the branch points of valence 2 crossed by these lines, that is points crossed or reached only by this line. We then sum the contributions over all these lines. For instance, if a line crosses only one branch point, the associated contribution will include two propagators $G_0=\hat{T}\left(1-\hat{T}\right)^{-1}$, one between the initial point and the branch point, one between the branch point and the final point plus the factors inserted at each point. Summing over all possible branch points crossed by a line yields the factor associated to the overall sum of single lines crossing the points Z_k:

$$\begin{aligned}&\hat{T}\left(1-\hat{T}\right)^{-1}\sum_{n\geqslant0}\int\prod_{l=1}^{n-1}\left\{\int\left(|\Psi(\theta-l_l,Z_l)|^2dZ_ldl_l\right)\hat{T}\left(1-\hat{T}\right)^{-1}\right\}|\Psi(\theta-l_n,Z_n)|^2\frac{\omega_0(J,\theta-l_n,Z_n)}{\bar{\mathcal{G}}_0(0,Z_n)}\\ =\;&\hat{T}\left(1-\hat{T}\right)^{-1}\frac{1}{1-|\Psi(\theta,Z)|^2\hat{T}\left(1-\hat{T}\right)^{-1}}|\Psi(\theta-l_n,Z_n)|^2\frac{\omega_0(J,\theta-l_n,Z_n)}{\bar{\mathcal{G}}_0(0,Z_n)}\\ =\;&\hat{T}\frac{1}{1-\left(1+|\Psi|^2\right)\hat{T}}|\Psi(\theta-l_n,Z_n)|^2\frac{\omega_0(J,\theta-l_n,Z_n)}{\bar{\mathcal{G}}_0(0,Z_n)}\end{aligned} \tag{325}$$

with $Z_0=X_1$ and $Z_{k+1}=X_2$ and $\prod_{l=1}^{0}$ is set to 1. The l_i are ranked such that: $l_1<\ldots<l_k$ We sum over all contributions of field insertions between (X_1,θ_1) and (X_2,θ_2) and integrate over the intermediate points. The factor $|\Psi|^2$ is seen as the operator multiplication by $|\Psi(\theta,Z)|^2$ at the point (θ,Z).

The sum (325) over the single lines is the Green function of the operator $1-\left(1+|\Psi|^2\right)\hat{T}$ with $\hat{T}$ and $|\Psi(\theta-l_n,Z_n)|^2\omega_0(J,\theta-l_n,Z_n)$ inserted at the starting and ending points. This quantity can be seen as a block $[(X_1,\theta_1),(X_2,\theta_2)]$.

6.2.2 Path integral formulation

Then, the series (324) can ultimately be rewritten as a sum over the number m of branch points (X_i,θ_i) with valence $k_i>2$: we draw all connected graphs whose vertices are the branch points $(X_1,\theta_1)...(X_m,\theta_m)$. We attach k_i blocks to the vertex (X_i,θ_i), the endpoint of one of them and the starting point of the others are fixed by the vertex. To each vertex, the factor $F((X_i,\theta_i))$ defined in (318) is associated. The extremities of the blocks that are not fixed are free and integrated over, except one of them which is equal to (Z,θ). Then the series (135) is the sum over m and over all types of graphs with m vertices.

Note that the sum of graph can be computed without ordering in time the fields. It amounts to replace (324) by:

$$\omega\left(\theta^{(i)},Z\right)-\omega\left(\theta^{(i)},Z\right)_{|\Psi|^2=0}=\int\frac{1}{n!}\sum_n\left(\frac{\delta^n\omega(J,\theta,Z)}{\prod\limits_{i=1}^n\delta|\Psi(\theta-l_i,Z_i)|^2}\right)_{|\Psi|^2=0}\prod_{i=1}^n|\Psi(\theta_i,Z_i)|^2\,d\theta_i dZ_i$$

As a consequence, the symetry factor of equivalent graphs factored by $\prod\limits_{i=1}^n|\Psi(\theta_i,Z_i)|^2$ and integrated over $\prod\limits_{i=1}^n d\theta_i dZ_i$ is:

$$\frac{1}{n!}\frac{n!}{\prod\limits_V k_V!}$$

where the product is over the vertices of valence k_V of the graph. The factor $n!$ comes from the exchange between the vertices $\prod\limits_{i=1}^n|\Psi(\theta_i,Z_i)|^2$ The $k_V!$ accounts for the exchange of the k_V vertices among the same graph.

The sum of lines connected by vertices can then be computed using the Green function $\frac{1}{1-\left(1+|\Psi|^2\right)\hat{T}}$ connecting the vertices of all possible valences.

As a consequence, the generating function for the graphs is equal to the partition function for an auxiliary complex field $\Lambda(X,\theta)$ with free Green function equal to $\frac{1}{1-\left(1+|\Psi|^2\right)\hat{T}}$ and interaction terms generating the various graphs with arbitrar vertices. The free part of the action for $\Lambda(X,\theta)$ is thus:

$$\int\Lambda(X,\theta)\left(1-\left(1+|\Psi|^2\right)\hat{T}\right)\Lambda^{\dagger}(X,\theta)\,d(X,\theta)$$

and the interaction terms have to induce the graphs with factor (318). The $k+2$ valence vertex, with $k\geqslant 1$

is thus described by a term involving (318) and writes:

$$\begin{aligned}
&\int \Lambda\left(Z^{(1)},\theta^{(1)}\right)\frac{\delta^{k}\left(\hat{T}\left(\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(2)}\right|}{c},Z^{(1)},Z^{(2)},\omega_{0}\right)\right)}{k!\prod_{l=3}^{k+2}\delta^{k}\omega_{0}\left(J,\theta^{(l)},Z^{(l)}\right)}\Lambda^{\dagger}\left(Z^{(2)},\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(2)}\right|}{c}\right)\\
&\times\prod_{l=3}^{k+2}\hat{T}\left(\left(Z^{(1)},\theta^{(1)}\right),\left(\theta^{(l)},Z^{(l)}\right)\right)\left(\Lambda^{\dagger}\left(\theta^{(l)},Z^{(l)}\right)\right)\prod_{l=1}^{k+2}d\left(\theta^{(l)},Z^{(l)}\right)\\
=&\int \Lambda\left(Z^{(1)},\theta^{(1)}\right)\frac{\delta^{k}\left(\hat{T}\left(\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(2)}\right|}{c},Z^{(1)},Z^{(2)},\omega_{0}\right)\right)}{k!\prod_{l=3}^{k+2}\delta^{k}\omega_{0}\left(J,\theta^{(l)},Z^{(l)}\right)}\Lambda^{\dagger}\left(Z^{(2)},\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(2)}\right|}{c}\right)\\
&\times\prod_{l=3}^{k+2}\hat{T}\left(\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(l)}\right|}{c},Z^{(1)},Z^{(l)},\omega_{0}\right)\Lambda^{\dagger}\left(\theta^{(l)},Z^{(l)}\right)d\theta^{(1)}\prod_{l=1}^{k+2}dZ^{(l)}
\end{aligned}$$

Having found the free part of the action and the required vertices, the sum of all graphs (324) yields, for $\frac{|\Psi(J,\theta_i,Z_i)|^2}{\mathcal{G}_0(0,Z_i)}\rightarrow|\Psi(J,\theta_i,Z_i)|^2$:

$$\begin{aligned}
&\omega_{0}(J,\theta,Z)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\int\hat{T}\Lambda^{\dagger}(Z,\theta)\int\prod_{i=1}^{n}\omega_{0}(J,\theta_{i},Z_{i})\left|\Psi(J,\theta_{i},Z_{i})\right|^{2}\Lambda(Z_{i},\theta_{i})\,d(Z_{i},\theta_{i})\exp(-S(\Lambda))\,\mathcal{D}\Lambda}{\exp(-S(\Lambda))\,\mathcal{D}\Lambda}\\
=&\ \omega_{0}(J,\theta,Z)+\frac{\int\hat{T}\Lambda^{\dagger}(Z,\theta)\exp\left(-S(\Lambda)+\int\Lambda(X,\theta)\,\omega_{0}(J,\theta,Z)\left|\Psi(J,\theta,Z)\right|^{2}d(X,\theta)\right)\mathcal{D}\Lambda}{\int\exp(-S(\Lambda))\,\mathcal{D}\Lambda}
\end{aligned}\tag{326}$$

with:

$$\begin{aligned}
S(\Lambda)=&\int\Lambda(X,\theta)\left(1-\left(1+|\Psi|^{2}\right)\hat{T}\right)\Lambda^{\dagger}(X,\theta)\,d(X,\theta)\\
&-\int\Lambda\left(Z^{(1)},\theta^{(1)}\right)\sum_{k}\frac{\delta^{k}\left(\hat{T}\left(\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(2)}\right|}{c},Z^{(1)},Z^{(2)},\omega_{0}\right)\right)}{k!\prod_{l=3}^{k+2}\delta^{k}\omega_{0}\left(J,\theta^{(l)},Z^{(l)}\right)}\Lambda^{\dagger}\left(Z^{(2)},\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(2)}\right|}{c}\right)\\
&\times\prod_{l=3}^{k+2}\hat{T}\left(\theta^{(1)}-\frac{\left|Z^{(1)}-Z^{(l)}\right|}{c},Z^{(1)},Z^{(l)},\omega_{0}\right)\Lambda^{\dagger}\left(\theta^{(l)},Z^{(l)}\right)d\theta^{(1)}\prod_{l=1}^{k+2}dZ^{(l)}\\
=&\int\Lambda(Z,\theta)\left(1-|\Psi|^{2}\hat{T}\right)\Lambda^{\dagger}(Z,\theta)\,d(Z,\theta)-\int\Lambda(Z,\theta)\,\hat{T}\left(\theta-\frac{\left|Z-Z^{(1)}\right|}{c},Z,Z^{(1)},\omega_{0}+\hat{T}\Lambda^{\dagger}\right)\\
&\times\Lambda^{\dagger}\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta
\end{aligned}$$

where:

$$\begin{aligned}
&\hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_{0}+\hat{T}\Lambda^{\dagger}\right)\\
=&\ \hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_{0}(Z,\theta)+\int\hat{T}\left(\theta-\frac{\left|Z-Z^{(1)}\right|}{c},Z^{(1)},Z,\omega_{0}\right)\Lambda^{\dagger}\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZ^{(1)}\right)
\end{aligned}$$

6.2.3 Saddle point approximation

The saddle point approximation yields the equations for $\Lambda^{\dagger}(Z,\theta)$ and $\Lambda(Z,\theta)$:

$$\left(\left(1-|\Psi|^{2}\hat{T}\right)\Lambda^{\dagger}\right)(Z,\theta)-\left(\hat{T}_{\omega_{0}+\hat{T}\Lambda^{\dagger}}\Lambda^{\dagger}\right)(Z,\theta)-\omega_{0}|\Psi|^{2}=0 \tag{327}$$

$$\Lambda(Z,\theta)=0$$

Using that:

$$\hat{T}\left(\omega_{0}+\hat{T}\Lambda^{\dagger}\right)\simeq\frac{\omega_{0}}{\omega_{0}+\hat{T}\Lambda^{\dagger}}\hat{T}$$

equation (327) writes:

$$\left(1-|\Psi|^{2}\hat{T}\right)\Lambda^{\dagger}-\frac{\omega_{0}}{\omega_{0}+\hat{T}\Lambda^{\dagger}}\hat{T}\Lambda^{\dagger}-\omega_{0}|\Psi|^{2}=0 \tag{328}$$

This can be rewritten as an equation for ω. Actually, using (326), under the saddle point approximation:

$$\omega(J,\theta,Z)=\omega_{0}(J,\theta,Z)+\hat{T}\Lambda^{\dagger}(Z,\theta) \tag{329}$$

and:

$$\hat{T}\Lambda^{\dagger}(Z,\theta)=\omega(J,\theta,Z)-\omega_{0}(J,\theta,Z)\equiv\Omega(J,\theta,Z)$$

so that (327) writes:

$$\Lambda^{\dagger}-(\Omega+\omega_{0})|\Psi|^{2}-\frac{\omega_{0}}{\omega_{0}+\Omega}\Omega=0 \tag{330}$$

Applying the operator $\hat{T}$ on the left leads to:

$$\Omega-\hat{T}(\Omega+\omega_{0})|\Psi|^{2}-\hat{T}\frac{\omega_{0}\Omega}{\omega_{0}+\Omega}=0 \tag{331}$$

Then using the expression for the background field:

$$\begin{aligned}\Psi(\theta,Z) &= \frac{\nabla_{\theta}\omega\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi_{0}|^{2}\right)}{U''(X_{0})\,\omega^{2}\left(J(\theta),\theta,Z,\mathcal{G}_{0}+|\Psi_{0}|^{2}\right)}\Psi_{0}(\theta,Z)\\ &= \frac{\nabla_{\theta}\omega}{U''(X_{0})\,\omega^{2}}\Psi_{0}(\theta,Z)\\ &= \frac{X_{0}\nabla_{\theta}\Omega}{U''(X_{0})(\omega_{0}+\Omega)^{2}}\end{aligned}$$

and:

$$|\Psi|^{2}=\frac{X_{0}^{2}}{U''(X_{0})}\frac{\nabla_{\theta}\Omega}{(\omega_{0}+\Omega)^{2}}$$

Equation (331) becomes:

$$\Omega-\hat{T}\frac{X_{0}^{2}}{U''(X_{0})}\frac{\nabla_{\theta}\Omega}{(\omega_{0}+\Omega)^{2}}\Omega-\hat{T}\frac{\omega_{0}\Omega}{\omega_{0}+\Omega}-\hat{T}\frac{X_{0}^{2}}{U''(X_{0})}\frac{\omega_{0}\nabla_{\theta}\Omega}{(\omega_{0}+\Omega)^{2}}=0$$

that is:

$$\Omega-\hat{T}\frac{\frac{X_{0}^{2}}{U''(X_{0})}\nabla_{\theta}\Omega+\omega_{0}\Omega}{\omega_{0}+\Omega}=0 \tag{332}$$

Remark that this equation is still valid for any background field related to ω by a relation of the type:

$$|\Psi|^{2}=f\left(\omega,\nabla_{\theta}^{l}\omega\right) \tag{333}$$

which yields:

$$\Omega-\hat{T}\omega f\left(\omega,\nabla_{\theta}^{l}\omega\right)-\hat{T}\frac{\omega_{0}(\omega-\omega_{0})}{\omega}=0$$

The second order expansion in derivatives of the right hand side of (332) yields a second order linear differential equation similar to the type of equation derived in the text.

6.2.4 Solution as function of external field and series expansion

We can also write a series expansion for the solution of (327) in terms of propagation functions for an external field $|\Psi|^2$. We write (327) as an operator equation:

$$\left(1 - |\Psi|^2 \hat{T} - \hat{T}_{\omega_0 + \hat{T}\Lambda^\dagger}\right) \Lambda^\dagger = \omega_0 |\Psi|^2$$

which leads to:

$$\begin{aligned}
\hat{T}\Lambda^\dagger &= \hat{T} \frac{1}{1 - |\Psi|^2 \hat{T} - \hat{T}_{\omega_0 + \hat{T}\Lambda^\dagger}} \omega_0 |\Psi|^2 \\
&= \hat{T} \frac{1}{1 - \left(1 + |\Psi|^2\right) \hat{T} - \left(\hat{T}_{\omega_0 + \hat{T}\Lambda^\dagger} - \hat{T}\right)} \omega_0 |\Psi|^2 \\
&\simeq \frac{\hat{T}}{1 - \left(1 + |\Psi|^2\right) \hat{T}} \frac{1}{1 - \left(\hat{T}_{\omega_0 + \hat{T}\Lambda^\dagger} - \hat{T}\right) \frac{1}{1 - \left(1 + |\Psi|^2\right)\hat{T}}} \omega_0 |\Psi|^2 \\
&= A \frac{1}{1 - \left(\hat{T}_{\omega_0 + \hat{T}\Lambda^\dagger} - \hat{T}\right) \hat{T}^{-1} A} \omega_0 |\Psi|^2 \qquad (334)
\end{aligned}$$

where:

$$A = \frac{\hat{T}}{1 - \left(1 + |\Psi|^2\right) \hat{T}} \qquad (335)$$

6.4.1 Recursive expansion of (334), first approximation

Equation (334) can be solved recursively, by expanding $\hat{T}_{\omega_0 + \hat{T}\Lambda^\dagger} - \hat{T}$ order by order:

$$\hat{T}\Lambda^\dagger = A \frac{1}{1 - \left(\hat{T}_{\omega_0 + A \frac{1}{1 - \left(\hat{T}_{\omega_0 + \hat{T}\Lambda^\dagger} - \hat{T}\right)\hat{T}^{-1} A} \omega_0 |\Psi|^2} - \hat{T}\right) \hat{T}^{-1} A} \omega_0 |\Psi|^2$$

and so on. In first approximation,the series expansion for $\hat{T}\Lambda^\dagger$ is:

$$\begin{aligned}
\hat{T}\Lambda^\dagger &= A \frac{1}{1 - \left(\hat{T}_{\omega_0 + A\omega_0 |\Psi|^2} - \hat{T}\right) \hat{T}^{-1} A} \omega_0 |\Psi|^2 \\
&\simeq A \frac{1}{1 - \frac{\omega_0}{\omega_0 + A\omega_0 |\Psi|^2} A} \omega_0 |\Psi|^2
\end{aligned}$$

or written in expanded form:

$$\omega = \omega_0 + \sum_{n \geqslant 0} \left(A \frac{\omega_0}{\omega_0 + A\omega_0 |\Psi|^2}\right)^n A\omega_0 |\Psi|^2 \qquad (336)$$

We use (335) to write A as:

$$A = \frac{\hat{T}}{1 - \left(1 + |\Psi|^2\right) \hat{T}} = \frac{1}{1 + |\Psi|^2} \frac{\left(1 + |\Psi|^2\right) \hat{T}}{1 - \left(1 + |\Psi|^2\right) \hat{T}}$$

Operator A is defined by successive convolutions, it can thus be approximated by:

$$A \simeq \frac{1}{1 + \left\langle |\Psi|^2 \right\rangle} \frac{\left(1 + \left\langle |\Psi|^2 \right\rangle\right) \hat{T}}{1 - \left(1 + \left\langle |\Psi|^2 \right\rangle\right) \hat{T}}$$

where $\frac{\left(1+\left\langle|\Psi|^2\right\rangle\right)\hat{T}}{1-\left(1+\left\langle|\Psi|^2\right\rangle\right)\hat{T}}$ can be computed as $\frac{\hat{T}}{1-\hat{T}}$ Given formulas (270), (274) and (276), it amounts to replace the constant C by $\left(1+\left\langle|\Psi|^2\right\rangle\right)C$ in the expansion of $\frac{\hat{T}}{1-\hat{T}}$. This modifies formula (276) by introducing a dependency in field inside the exponential function. We thus have:

$$A\left(\left(Z,\theta\right),\left(Z_1,\theta-l_1\right)\right)\simeq\frac{\exp\left(-cl_1-\alpha\left(1+\left\langle|\Psi|^2\right\rangle\right)\left(\left(cl_1\right)^2-\left|Z-Z_1\right|^2\right)\right)}{\left(1+\left\langle|\Psi|^2\right\rangle\right)B}H\left(cl_1-\left|Z-Z_1\right|\right) \tag{337}$$

Inserting formula (337) in the expression (336) for ω leads to:

$$\begin{aligned}\omega\left(Z,\theta\right) &= \omega_0\left(J,\theta,Z\right) \qquad (338)\\ &+\int\sum_{k=0}^{\infty}\frac{\exp\left(-c\sum_{i=0}^{k}l_i-\alpha\left(1+\left\langle|\Psi|^2\right\rangle\right)\left(\sum_{i=0}^{k}\left(cl_i\right)^2-\sum_{l=0}^{k-1}\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{B^{k+1}}\\ &\times\prod_{i=1}^{k}\left(\frac{\omega_0\left(\theta-l_i,Z_i\right)}{\omega_0\left(\theta-l_i,Z_i\right)+A\omega_0\left|\Psi\right|^2\left(\theta-l_i,Z_i\right)}\right)\frac{\omega_0\left(J,\theta-l_k,Z_k\right)}{\left(1+\left\langle|\Psi|^2\right\rangle\right)}\left|\Psi\left(\theta-l_k,Z_k\right)\right|^2dZ_idl_i\end{aligned}$$

and:

$$\begin{aligned}\omega^{-1}\left(Z,\theta\right) &= \omega_0^{-1}\left(J,\theta,Z\right) \qquad (339)\\ &+\frac{G'\left[J,\omega,\theta,Z,\Psi\right]}{F'\left[J,\omega,\theta,Z,\Psi\right]}\int\sum_{k=0}^{\infty}\frac{\exp\left(-c\sum_{i=0}^{k}l_i-\alpha\left(1+\left\langle|\Psi|^2\right\rangle\right)\left(\sum_{i=0}^{k}\left(cl_i\right)^2-\sum_{l=0}^{k-1}\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{D^{k+1}}\\ &\times\prod_{i=1}^{k}\left(\frac{\omega_0\left(\theta-l_i,Z_i\right)}{\omega_0\left(\theta-l_i,Z_i\right)+A\omega_0\left|\Psi\right|^2\left(\theta-l_i,Z_i\right)}\right)\frac{\omega_0^{-1}\left(J,\theta-l_k,Z_k\right)}{\left(1+\left\langle|\Psi|^2\right\rangle\right)}\left|\Psi\left(\theta-l_k,Z_k\right)\right|^2dZ_idl_i\end{aligned}$$

6.4.2 Recursive expansion of (334), full series expansion

Formula (338) can be refined by including the full series expansion of (334). In a first step we can approximate:

$$\begin{aligned}&\frac{1}{1-\left(1+|\Psi|^2\right)\hat{T}}\frac{1}{1-\left(\hat{T}_{\omega_0+\hat{T}\Lambda^{\dagger}}-\hat{T}\right)\frac{1}{1-\left(1+|\Psi|^2\right)\hat{T}}}\\ \simeq\;&\frac{1}{1-\left(1+|\Psi|^2\right)\hat{T}}\frac{1}{1-\frac{\frac{\delta\hat{T}}{\delta\omega_0}}{\hat{T}}\hat{T}\Lambda^{\dagger}\frac{\hat{T}}{1-\left(1+|\Psi|^2\right)\hat{T}}}\end{aligned}$$

so that:

$$\begin{aligned}\hat{T}\Lambda^{\dagger} &= A\frac{1}{1-\frac{\frac{\delta\hat{T}}{\delta\omega_0}}{\hat{T}}\hat{T}\Lambda^{\dagger}A}\omega_0\left|\Psi\right|^2\\ &= A\sum_{n\geqslant0}\left(\frac{\frac{\delta\hat{T}}{\delta\omega_0}}{\hat{T}}\hat{T}\Lambda^{\dagger}A\right)^n\omega_0\left|\Psi\right|^2\\ &= A\sum_{k\geqslant0}\left(\left(\frac{\delta\hat{T}}{\delta\omega_0}\left(A\sum_{l\geqslant0}\left(\left(\frac{\delta\hat{T}}{\delta\omega_0}\hat{T}\Lambda^{\dagger}\right)A\right)^l\omega_0\left|\Psi\right|^2\right)\right)A\right)^k\omega_0\left|\Psi\right|^2=...\end{aligned}$$

The iteration of the previous relation amounts to the successive action of the operator:

$$\begin{aligned} O = &\int d\left(Z,\theta\right) \frac{\frac{\delta \hat{T}}{\delta \omega_0}}{\hat{T}} \left(Z,\theta\right) \left(\int A\left(\left(Z,\theta\right),\left(Z',\theta'\right)\right) \omega_0 \left|\Psi\right|^2 \left(Z',\theta'\right) d\left(Z',\theta'\right)\right) \\ &\times \int d\left(Z,\theta\right)_1 A\left(\left(Z,\theta\right),\left(Z,\theta\right)_1\right) \omega_0 \left|\Psi\right|^2 \left(Z_1,\theta_1\right) \frac{\delta}{\delta \omega_0 \left|\Psi\right|^2 \left(Z,\theta\right)} \end{aligned} \tag{340}$$

where $\frac{\delta}{\delta\omega_0|\Psi|^2(Z,\theta)}$ does not act on $A\left(\left(Z,\theta\right),\left(Z,\theta\right)_1\right)$. As a consequence:

$$\hat{T}\Lambda^{\dagger} = \sum_{n\geqslant 0} O^n \omega_0 A \left|\Psi\right|^2 = \frac{1}{1-O} \omega_0 A \left|\Psi\right|^2$$

with $\frac{1}{n!}$ accounting for the permutations between the powers of $\left|\Psi\right|^2$. Setting $(Z,\theta)=X$ the series expansion of $\hat{T}\Lambda^{\dagger}$ can be obtained as:

$$\begin{aligned} \hat{T}\Lambda^{\dagger} = & \sum_{l_1,\dots,l_n,\sum_{i=1}^{n} l_i = n-1} \; \sum_{\substack{f\in\{1,\dots,n\}^{\{1,\dots,n\}},\\ f(k)\notin\left\{\sum_{i=1}^{k-1} l_i+1,\dots \sum_{i=1}^{k} l_i\right\}}} A\left(X,\hat{X}_1\right)\dots A\left(\hat{X}_{l_1},X_0\right) A\left(X_0,X_1'\right)\omega_0\left|\Psi\right|^2\left(X_1'\right)dX_1' \\ & \times \prod_{k=1}^{n}\frac{1}{\sharp k!} \times \prod_{k=2}^{n} A\left(X_k,\hat{X}_{\sum_{i=1}^{k-1} l_i+1}\right)\dots A\left(X_{l_1},\hat{X}_{\sum_{i=1}^{k} l_i}\right) A\left(\hat{X}_{\sum_{i=1}^{k} l_i},X_k'\right)\omega_0\left|\Psi\right|^2\left(X_k'\right)dX_k' \\ & \times \prod_{k=2}^{n} \frac{\frac{\delta \hat{T}}{\delta \omega_0}}{\hat{T}} \left(X_k\right) \delta\left(X_k - \hat{X}_{f(k)}\right) \end{aligned} \tag{341}$$

with the convention that for $l_i=0$ the successive products of convolution reduces to:

$$A\left(X_k,X_k'\right)\omega_0\left|\Psi\right|^2\left(X'\right)dX'$$

The factors $\sharp k$ are defined by:

$$\sharp k = \sum_{l_i=1}^{n} \delta_{l_i,k}$$

The series expansion can be written in a more compact way:

$$\begin{aligned} \hat{T}\Lambda^{\dagger} = & \sum_n \sum_{l_1,\dots,l_n,\sum_{i=1}^{n} l_i = n-1} \; \sum_{\substack{f\in\{1,\dots,n\}^{\{1,\dots,n\}},\\ f(k)\notin\left\{\sum_{i=1}^{k-1} l_i+1,\dots \sum_{i=1}^{k} l_i\right\}}} A^{l_1}\left(X,\hat{X}_1,\dots,\hat{X}_{l_1},X_0,X_1'\right)\omega_0\left|\Psi\right|^2\left(X_1'\right)dX_1' \\ & \times \prod_{k=1}^{n}\frac{d\hat{X}_k}{\sharp k!} \times \prod_{k=2}^{n} A^{l_k}\left(X_k,\hat{X}_{\sum_{i=1}^{k-1} l_i+1},\dots,\hat{X}_{\sum_{i=1}^{k} l_i},X_k'\right)\omega_0\left|\Psi\right|^2\left(X_k'\right)dX_k' \times \prod_{k=2}^{n} \frac{\frac{\delta \hat{T}}{\delta \omega_0}}{\hat{T}} \left(X_k\right) \delta\left(X_k - \hat{X}_{f(k)}\right) \end{aligned} \tag{342}$$

with:

$$A^{l_k}\left(X_k,\hat{X}_{\sum_{i=1}^{k-1} l_i+1},\dots,\hat{X}_{\sum_{i=1}^{k} l_i},X_k'\right) = A\left(X_k,\hat{X}_{\sum_{i=1}^{k-1} l_i+1}\right)\dots A\left(X_{l_1},\hat{X}_{\sum_{i=1}^{k} l_i}\right) A\left(\hat{X}_{\sum_{i=1}^{k} l_i},X_k'\right)$$

In a second step, all the derivatives $\frac{\frac{\delta^l \hat{T}}{\delta^l \omega_0}}{\hat{T}}$ can be included and yield the series expansion:

$$\hat{T}\Lambda^{\dagger} = \sum_{n,p} \sum_{\sum_{i=1}^{p} r_i = n-1} \sum_{l_1,\ldots,l_n, \sum_{i=1}^{n} l_i = p} \sum_{\substack{f \in \{1,\ldots,p\}^{\{1,\ldots,n\}}, \\ f(k) \notin \left\{ \sum_{i=1}^{k-1} l_i + 1, \ldots \sum_{i=1}^{k} l_i \right\}}} A^{l_1}\left(X, \hat{X}_1, \ldots, \hat{X}_{l_1}, X_0, X_1'\right) \omega_0 \left|\Psi\right|^2 \left(X_1'\right) dX_1'$$
$$\times \prod_{k=1}^{p} \frac{d\hat{X}_k}{\sharp k!} \times \prod_{k=2}^{n} A^{l_k}\left(X_k, \hat{X}_{\sum_{i=1}^{k-1} l_i + 1}, \ldots, \hat{X}_{\sum_{i=1}^{k} l_i}, X_k'\right) \omega_0 \left|\Psi\right|^2 \left(X_k'\right) dX_k'$$
$$\times \prod_{i}^{p} \frac{\frac{\delta^{r_i} \hat{T}}{\delta^{r_i} \omega_0 \left(\hat{X}_i\right)}}{\hat{T}} \prod_{k=2}^{n} \delta\left(X_k - \hat{X}_{f(k)}\right) \tag{343}$$

The propagators $A^{l_k}\left(X_k, \hat{X}_{\sum_{i=1}^{k-1} l_i + 1}, \ldots, \hat{X}_{\sum_{i=1}^{k} l_i}, X_k'\right)$ can be computed using (337):

$$A = \frac{\hat{T}}{1 - \left(1 + \left|\Psi\right|^2\right)\hat{T}} = \frac{1}{1 + \left|\Psi\right|^2} \frac{\left(1 + \left|\Psi\right|^2\right)\hat{T}}{1 - \left(1 + \left|\Psi\right|^2\right)\hat{T}}$$

Operator A being defined by successive convolutions, it can be approximated by:

$$\frac{1}{1 + \left\langle \left|\Psi\right|^2 \right\rangle} \frac{\left(1 + \left\langle \left|\Psi\right|^2 \right\rangle\right)\hat{T}}{1 - \left(1 + \left\langle \left|\Psi\right|^2 \right\rangle\right)\hat{T}}$$

and $\frac{\left(1 + \left\langle \left|\Psi\right|^2 \right\rangle\right)\hat{T}}{1 - \left(1 + \left\langle \left|\Psi\right|^2 \right\rangle\right)\hat{T}}$ can be computed as $\frac{\hat{T}}{1 - \hat{T}}$ Given formulas (270), (274) and (276), it amounts to replace the constant C by $\left(1 + \left\langle \left|\Psi\right|^2 \right\rangle\right) C$. This modifies (276) by introducing a dependency in field inside the exponential function:

$$A\left(\left(Z, \theta\right), \left(Z_1, \theta - l_1\right)\right) \simeq \frac{\exp\left(-cl_1 - \alpha\left(1 + \left\langle \left|\Psi\right|^2 \right\rangle\right)\left(\left(cl_1\right)^2 - \left|Z - Z_1\right|^2\right)\right)}{B} H\left(cl_1 - \left|Z - Z_1\right|\right) \tag{344}$$

In (342) the vertices $\frac{\frac{\delta \hat{T}}{\delta \omega_0}}{\hat{T}}\left(X_k\right)$ and the ω_0 arising in factor can be approximated by the average $\frac{1}{\langle \omega_0 \rangle}$ and $\langle \omega_0 \rangle$, so that (342) writes:

$$\Omega = \hat{T}\Lambda^{\dagger} = \sum_{n} \sum_{l_1,\ldots,l_n, \sum_{i=1}^{n} l_i = n-1} \sum_{\substack{f \in \{1,\ldots,n\}^{\{1,\ldots,n\}}, \\ f(k) \notin \left\{ \sum_{i=1}^{k-1} l_i + 1, \ldots \sum_{i=1}^{k} l_i \right\}}} A^{l_1}\left(X, \hat{X}_1, \ldots, \hat{X}_{l_1}, X_0, X_1'\right) \left|\Psi\right|^2 \left(X_1'\right) dX_1'$$
$$\times \prod_{k=1}^{n} \frac{d\hat{X}_k}{\sharp k!} \times \prod_{k=2}^{n} A^{l_k}\left(X_k, \hat{X}_{\sum_{i=1}^{k-1} l_i + 1}, \ldots, \hat{X}_{\sum_{i=1}^{k} l_i}, X_k'\right) \left|\Psi\right|^2 \left(X_k'\right) dX_k' \times \prod_{k=2}^{n} \delta\left(X_k - \hat{X}_{f(k)}\right) \tag{345}$$

6.2.5 Corrections to the saddle point

The corrections to the saddle point approximation are obtained by expanding $S\left(\Lambda, \Lambda_0^{\dagger} + \Lambda^{\dagger}\right)$ around the solution of (327) $\Lambda_0^{\dagger}$. Given that:

$$S\left(\Lambda,\Lambda_0^\dagger+\Lambda^\dagger\right)\simeq\int\Lambda\left(Z,\theta\right)\left(1-\left|\Psi\right|^2\hat{T}\right)\Lambda^\dagger\left(Z,\theta\right)d\left(Z,\theta\right)$$
$$-\int\Lambda\left(Z,\theta\right)\hat{T}\left(\theta-\frac{\left|Z-Z^{(1)}\right|}{c},Z,Z^{(1)},\omega_0+\hat{T}\Lambda_0^\dagger\right)\Lambda^\dagger\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta$$
$$-\int\Lambda\left(Z,\theta\right)\left(\hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_0+\hat{T}\left(\Lambda_0^\dagger+\Lambda^\dagger\right)\right)-\hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_0+\hat{T}\Lambda_0^\dagger\right)\right)$$
$$\times\Lambda_0^\dagger\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta$$

It yields the second order corrections to S:

$$\begin{aligned}
&\hat{S}\left(\Lambda^\dagger,\Lambda\right)\\
=\;&S\left(\Lambda,\Lambda_0^\dagger+\Lambda^\dagger\right)-S\left(0,\Lambda_0^\dagger\right)\\
\simeq\;&\int\Lambda\left(Z,\theta\right)\left(1-\left|\Psi\right|^2\hat{T}\right)\Lambda^\dagger\left(Z,\theta\right)d\left(Z,\theta\right)\\
&-\int\Lambda\left(Z,\theta\right)\hat{T}\left(\theta-\frac{\left|Z-Z^{(1)}\right|}{c},Z,Z^{(1)},\omega_0+\hat{T}\Lambda_0^\dagger\right)\Lambda^\dagger\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta\\
&-\int\Lambda\left(Z,\theta\right)\left(\frac{\delta}{\delta\omega_0\left(Z,\theta\right)}\hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_0+\hat{T}\Lambda_0\right)\right)\left(\hat{T}\Lambda^\dagger\right)\left(Z,\theta\right)\\
&\times\Lambda_0^\dagger\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta\\
&-\frac{1}{2}\int\Lambda\left(Z,\theta\right)\left(\frac{\delta^2}{\delta\omega_0^2\left(Z,\theta\right)}\hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_0+\hat{T}\Lambda_0\right)\right)\\
&\times\left(\left(\hat{T}\Lambda^\dagger\right)\left(Z,\theta\right)\right)^2\Lambda_0^\dagger\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta
\end{aligned}$$

and the quadratic expansion of $\hat{S}$ is:

$$\hat{S}\left(\Lambda^\dagger,\Lambda\right)\simeq\int\Lambda\left(Z,\theta\right)\left(1-\left|\Psi\right|^2\hat{T}_{\omega_0}-\hat{T}_{\omega_0+\hat{T}\Lambda_0^\dagger}-\frac{\delta\hat{T}_{\omega_0+\hat{T}\Lambda_0}}{\delta\omega_0\left(Z,\theta\right)}\Lambda_0^\dagger\hat{T}_{\omega_0}\right)\Lambda^\dagger\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta$$

The corrections to $\Lambda_0^\dagger$ are computed using this second order expansion of:

$$\int\hat{T}\Lambda^\dagger\left(Z,\theta\right)\exp\left(-S\left(\Lambda\right)+\int\Lambda\left(X,\theta\right)\omega_0\left(J,\theta,Z\right)\left|\Psi\left(J,\theta,Z\right)\right|^2d\left(X,\theta\right)\right)\mathcal{D}\Lambda$$

around $\Lambda=0$ and $\Lambda^\dagger=\Lambda_0^\dagger$ as well as similar expansion of $\int\exp\left(-S\left(\Lambda\right)\right)\mathcal{D}\Lambda$ around its minimum $\Lambda=\Lambda^\dagger=0$. Given that all correlations function with a different number of $\Lambda^\dagger$ and Λ is null, including the first order corrections yield the following expression for ω:

$$\begin{aligned}
\omega\;=\;&\omega_0 \qquad (346)\\
&+\frac{1}{\int\exp\left(-S\left(\Lambda\right)\right)\mathcal{D}\Lambda}\hat{T}\Lambda_0^\dagger\int\exp\left(-\int\Lambda\left(Z,\theta\right)\left(1-\left(1+\left|\Psi\right|^2\right)\hat{T}_{\omega_0}+\left(1-\hat{T}_{\omega_0+\hat{T}\Lambda_0^\dagger}-\frac{\delta\hat{T}_{\omega_0+\hat{T}\Lambda_0}}{\delta\omega_0\left(Z,\theta\right)}\Lambda_0^\dagger\hat{T}_{\omega_0}\right)\right)\right.\\
&\left.\times\Lambda^\dagger\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta\right)\mathcal{D}\Lambda
\end{aligned}$$

The factor $\int \exp\left(-S\left(\Lambda\right)\right)\mathcal{D}\Lambda$ is computed by expanding around $\Lambda_0=\Lambda_0^{\dagger}=0$:

$$\begin{aligned}\int \exp\left(-S\left(\Lambda\right)\right)\mathcal{D}\Lambda &\simeq \int \exp\left(-\int \Lambda\left(X,\theta\right)\left(1-\left(1+\left|\Psi\right|^2\right)\hat{T}\right)\Lambda^{\dagger}\left(X,\theta\right)d\left(X,\theta\right)\right)\mathcal{D}\Lambda \\ &= \det\left(1-\left(1+\left|\Psi\right|^2\right)\hat{T}\right)^{-1}\end{aligned}$$

and the equation (329) is replaced by:

$$\begin{aligned}\omega &\simeq \omega_0+\hat{T}\Lambda_0^{\dagger}\frac{\left(\det\left(1-\left(1+\left|\Psi\right|^2\right)\hat{T}+\left(1-\hat{T}_{\omega_0+\hat{T}\Lambda_0^{\dagger}}-\frac{\delta\hat{T}_{\omega_0+\hat{T}\Lambda_0}}{\delta\omega_0(Z,\theta)}\Lambda_0^{\dagger}\hat{T}_{\omega_0}\right)\right)\right)^{-1}}{\det\left(1-\left(1+\left|\Psi\right|^2\right)\hat{T}\right)^{-1}} \qquad (347)\\ &= \omega_0+\hat{T}\Lambda_0^{\dagger}\left(\det\left(1+\left(1-\hat{T}_{\omega_0+\hat{T}\Lambda_0^{\dagger}}-\frac{\delta\hat{T}_{\omega_0+\hat{T}\Lambda_0}}{\delta\omega_0\left(Z,\theta\right)}\Lambda_0^{\dagger}\hat{T}_{\omega_0}\right)\left(1-\left(1+\left|\Psi\right|^2\right)\hat{T}\right)^{-1}\right)\right)^{-1}\end{aligned}$$

Using that $\hat{T}_{\omega_0+\hat{T}\Lambda_0^{\dagger}}$ is of order $\frac{1}{\omega_0+\hat{T}\Lambda^{\dagger}}$ and:

$$\hat{T}\left(\omega_0+\hat{T}\Lambda^{\dagger}\right)\simeq\frac{\omega_0}{\omega_0+\hat{T}\Lambda^{\dagger}}\hat{T}$$

we have:

$$\frac{\delta\hat{T}_{\omega_0+\hat{T}\Lambda_0}}{\delta\omega_0\left(Z,\theta\right)}\simeq-\frac{\omega_0}{\left(\omega_0+\hat{T}\Lambda^{\dagger}\right)^2}\hat{T}$$

and the factor $-\hat{T}_{\omega_0+\hat{T}\Lambda_0^{\dagger}}-\frac{\delta\hat{T}_{\omega_0+\hat{T}\Lambda_0}}{\delta\omega_0(Z,\theta)}\Lambda_0^{\dagger}\hat{T}_{\omega_0}$ arising in (346) writes:

$$\begin{aligned}-\hat{T}_{\omega_0+\hat{T}\Lambda_0^{\dagger}}-\frac{\delta\hat{T}_{\omega_0+\hat{T}\Lambda_0}}{\delta\omega_0\left(Z,\theta\right)}\Lambda_0^{\dagger}\hat{T} &= -\frac{\omega_0}{\omega_0+\hat{T}\Lambda_0^{\dagger}}\hat{T}+\frac{\omega_0\hat{T}\Lambda_0^{\dagger}}{\left(\omega_0+\hat{T}\Lambda_0^{\dagger}\right)^2}\hat{T} \\ &= \frac{\omega_0^2}{\left(\omega_0+\hat{T}\Lambda_0^{\dagger}\right)^2}\hat{T}\end{aligned}$$

and (347) becomes:

$$\omega=\omega_0+\hat{T}\Lambda_0^{\dagger}\left(\det\left(1+\left(1+\frac{\omega_0\left(2\omega_0+\hat{T}\Lambda_0^{\dagger}\right)}{\left(\omega_0+\hat{T}\Lambda_0^{\dagger}\right)^2}\hat{T}\left(1+\left|\Psi\right|^2\right)\hat{T}\right)^{-1}\right)\right)^{-1} \qquad (348)$$

Equation (113) is considered along with the defining equation (328) for $\Lambda_0^{\dagger}$:

$$\Lambda_0^{\dagger}-\left(\left|\Psi\right|^2+\frac{\omega_0}{\omega_0+\hat{T}\Lambda_0^{\dagger}}\right)\hat{T}\Lambda_0^{\dagger}-\omega_0\left|\Psi\right|^2=0 \qquad (349a)$$

and the relation (333) between background field and frequency:

$$\left|\Psi\right|^2=f\left(\omega,\nabla_{\theta}^{l}\omega\right)$$

We set again $\Omega=\omega-\omega_0$ and we have:

$$\hat{T}\Lambda_0^{\dagger}\simeq\Omega\det\left(1+\left(1+\frac{\omega_0\left(2\omega_0+\Omega\right)}{\left(\omega_0+\Omega\right)^2}\hat{T}\left(1+f\left(\omega,\nabla_{\theta}^{l}\omega\right)\right)\hat{T}\right)^{-1}\right)$$

so that applying $\hat{T}$ to (349a) yields:

$$\begin{aligned}
&\Omega \det\left(1+\left(1+\frac{\omega_0\left(2\omega_0+\Omega\right)}{\left(\omega_0+\Omega\right)^2}\hat{T}\left(1+f\left(\omega,\nabla_\theta^l\omega\right)\right)\hat{T}\right)^{-1}\right) \\
=\ & \hat{T}\left(\left(f\left(\omega,\nabla_\theta^l\omega\right)+\frac{\omega_0}{\omega_0+\Omega\det\left(1+\left(1+\frac{\omega_0(2\omega_0+\Omega)}{(\omega_0+\Omega)^2}\hat{T}\left(1+f\left(\omega,\nabla_\theta^l\omega\right)\right)\hat{T}\right)^{-1}\right)}\right)\right. \\
&\left.\times\Omega \det\left(1+\left(1+\frac{\omega_0\left(2\omega_0+\Omega\right)}{\left(\omega_0+\Omega\right)^2}\hat{T}\left(1+f\left(\omega,\nabla_\theta^l\omega\right)\right)\hat{T}\right)^{-1}\right)+\omega_0 f\left(\omega,\nabla_\theta^l\omega\right)\right)
\end{aligned}$$

6.2.6 Time dependent background field

For non constant background fields, the link between $\Psi(\theta,Z)$ and $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ is:

$$\Psi=\delta\Psi+\Psi_0$$

and thus:

$$\begin{aligned}
\Psi(\theta,Z) &= \left(\frac{\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)}{U''(X_0)-\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\right)}\right)\Psi_0(\theta,Z)+\Psi_0(\theta,Z) \\
&\simeq \left(\frac{\nabla_\theta\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\omega^2\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)U''(X_0)-\nabla_\theta\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}\right)\Psi_0(\theta,Z)+\Psi_0(\theta,Z)
\end{aligned}$$

$$\Psi^\dagger(\theta,Z)=\Psi_0^\dagger(\theta,Z)$$

$$|\Psi(\theta,Z)|^2\simeq|\Psi_0(\theta,Z)|^2+\Psi_0^\dagger(\theta,Z)\left(\frac{\nabla_\theta\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{\omega^2\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)U''(X_0)-\nabla_\theta\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}\right)\Psi_0(\theta,Z)$$

Equation (331) leads to:

$$\Omega-\hat{T}(\Omega+\omega_0)\left(|\Psi_0(\theta,Z)|^2+\Psi_0^\dagger(\theta,Z)\frac{\nabla_\theta(\omega_0+\Omega)}{(\omega_0+\Omega)^2U''(X_0)-\nabla_\theta(\omega_0+\Omega)}\Psi_0(\theta,Z)\right)-\hat{T}\frac{\omega_0\Omega}{\omega_0+\Omega}=0$$

At the lowest order, this becomes:

$$\Omega-\hat{T}(\Omega+\omega_0)|\Psi_0(\theta,Z)|^2-\hat{T}\frac{\omega_0\Omega}{\omega_0+\Omega}=0$$

$$\Omega-\hat{T}\Omega|\Psi_0(\theta,Z)|^2-\hat{T}\frac{\omega_0\Omega}{\omega_0+\Omega}=\hat{T}\left(\omega_0|\Psi_0(\theta,Z)|^2\right)$$

for $\Omega << \omega_0$, this becomes:

$$\left(1-\hat{T}\left(1+|\Psi_0(\theta,Z)|^2\right)\right)\Omega=\hat{T}\left(\omega_0|\Psi_0(\theta,Z)|^2\right)$$

with solution:

$$\begin{aligned}
\Omega &= \frac{1}{1-\hat{T}\left(1+|\Psi_0(\theta,Z)|^2\right)}\hat{T}\left(\omega_0|\Psi_0(\theta,Z)|^2\right) \qquad (350)\\
&= \frac{1}{1-\frac{-\hat{T}|\Psi_0(\theta,Z)|^2}{1-T}}\frac{\hat{T}}{1-\hat{T}}\omega_0|\Psi_0(\theta,Z)|^2
\end{aligned}$$

The operator $\frac{\hat{T}}{1-\hat{T}}$ summing over all lines between two given points has been estimated in Appendix 5. Between two points (Z_1, θ_1) and (Z_2, θ_2) the factor associated to the sum of paths is of order:

$$\frac{\hat{T}}{1-\hat{T}}\left((Z_1,\theta_1),(Z_2,\theta_2)\right) = \frac{\exp\left(-c(\theta_2-\theta_1)-\alpha\left(c^2(\theta_2-\theta_1)^2-|Z_2-Z_1|^2\right)\right)}{A} H\left(\theta_2-\theta_1-\frac{|Z_2-Z_1|}{c}\right) \tag{351}$$

with α and A are some parameters and H is the heaviside function. The insertion of the field in (350) yields:

$$\begin{aligned}\Omega &= \int\sum_{k=0}^{\infty}\frac{\exp\left(-\alpha\left(\left(1+\frac{c}{\alpha}\right)(\theta_2-\theta_1)-\sum_{l=0}^{k}\frac{|Z_l-Z_{l+1}|}{c}\right)\right)}{A^{k+1}} \\ &\times\left(\prod_{l=1}^{k}\int|\Psi(\theta-l_l,Z_l)|^2 dZ_l dl_l\right)\omega_0(J,\theta-l_k,Z_k)|\Psi(\theta-l_k,Z_k)|^2\end{aligned} \tag{352}$$

This value of Ω represents the fluctuations in frequencies due to the time dependency in potential. In first approximation, this combines with

6.3 Extension: Excitatory vs inhibitory interaction

6.3.1 Series expansion for the frequencies

The method of section 6.2 can be extended straightforwarly in the case of two types of interactions. We will derive a path integral description for the frequencies.

We consider n populations, each caracterized by their frequencies $i = 1, ..., n$. They interact positively or negatively. Each population is defined by a field Ψ_i and freqncies $\omega_i(\theta, Z)$. Equations for frequencies are defined by:

$$\begin{aligned}\omega_i(\theta,Z) &= F_i\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_1)\frac{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega_i(\theta,Z)}G^{ij}\right. \\ &\left.\times W\left(\frac{\omega_i(\theta,Z)}{\omega_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}\right)\left(\bar{\mathcal{G}}_{0j}(0,Z_1)+\left|\Psi_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned} \tag{353}$$

For example, if $i, j = 1, 2$, a matrix g of the form:

$$G = \begin{pmatrix} 1 & -g \\ -g & 1 \end{pmatrix}$$

represents inhibitory interactions between the two populations. More generally, the matrix G is $n \times n$ with coefficients in the interval $[-1, 1]$. The sum over indices is understood for j. The resolution of (353) follows the same principle as for (306), with a n components vector of frequencies $\omega(J, \theta, Z)$. Writing the series expansion for $\omega(J, \theta, Z)$:

$$\begin{aligned}\omega(J,\theta,Z) &= \sum_r\left(\frac{\delta^r\omega(J,\theta,Z)}{\prod_{i=1}^{r}\delta|\Psi(\theta-l_i,Z_i)|^2}\right)_{|\Psi|^2=0}\prod_{i=1}^{r}|\Psi(\theta-l_i,Z_i)|^2 \\ &= \left(\sum_{m=1}^{n}\sum_{i=1}^{m}\sum_{(line_1,...,line_m)}\prod_i F(line_i)\prod_B F(B)\right)\prod_{i=1}^{n}|\Psi(\theta-l_i,Z_i)|^2\end{aligned} \tag{354}$$

where:

$$\left(\frac{\delta^{r}\omega\left(J,\theta,Z\right)}{\prod\limits_{i=1}^{r}\delta\left|\Psi\left(\theta-l_{i},Z_{i}\right)\right|^{2}}\right)_{|\Psi|^{2}=0} \tag{355}$$

and:

$$\left|\Psi\left(\theta-l_{i},Z_{i}\right)\right|^{2}$$

are considered as $(1,r)$ and $(1,0)$ tensors respectively, the expansion of the first order derivative is similar to (264) and is given by:

$$\begin{aligned}\left(\frac{\delta\omega\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_{1},Z_{1}\right)\right|^{2}}\right)_{|\Psi|^{2}=0} &= \sum_{n=1}^{\infty}\int\prod_{l=1}^{n}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_{0},0\right)\\&\times\Omega_{0}\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_{1}\right)\times\delta\left(l_{1}-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c}\right)\prod_{l=1}^{n-1}dZ^{(l)}\end{aligned}$$

with ω_0 a n component vector describing a solution for $|\Psi|^2=0$ and $\Omega_0\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)$ is a diagonal matrix with components $\omega_{0i}\left(J,\theta-\sum_{l=1}^{n}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_1\right)$.

For practical purposes, we also define the diagonal matrix $D\left(|\Psi|^2\right)$ with $|\Psi_i|^2$ on the diagonal. More generally, for any expression $H\left(\omega_{0i},|\Psi_i|^2\right)$, we define $D\left(H\left(\omega_0,|\Psi|^2\right)\right)$ the diagonal matrix with components $H\left(\omega_{0i},|\Psi_i|^2\right)$.

The quantity $\Omega(J,\theta,Z)|\Psi|^2$ is a vector with components $\omega_i(J,\theta,Z)|\Psi_i|^2$. The expressions $\left(\frac{\delta\omega(J,\theta,Z)}{\delta|\Psi(\theta-l_1,Z_1)|^2}\right)_{|\Psi|^2=0}$ and $\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_0,0\right)$ are $n\times n$ matrices:

$$\left(\left(\frac{\delta\omega\left(J,\theta,Z\right)}{\delta\left|\Psi\left(\theta-l_{1},Z_{1}\right)\right|^{2}}\right)_{|\Psi|^{2}=0}\right)_{ij}=\left(\frac{\delta\omega_{i}\left(J,\theta,Z\right)}{\delta\left|\Psi_{j}\left(\theta-l_{1},Z_{1}\right)\right|^{2}}\right)_{|\Psi|^{2}=0}$$

and:

$$\begin{aligned}&\hat{T}_{ij}\left(\theta,Z,Z_{1}\omega,\Psi\right)\\=&\frac{G^{ij}\frac{\kappa}{N}\omega_{i}\left(J,\theta,Z\right)T\left(Z,Z_{1}\right)F_{i}'\left[J,\omega,\theta,Z,\Psi\right]}{\omega_{i}^{2}\left(J,\theta,Z\right)+G^{ij}\left(\int\frac{\kappa}{N}\omega_{j}\left(J,\theta-\frac{|Z-Z'|}{c},Z'\right)\left(\bar{\mathcal{G}}_{0j}\left(0,Z'\right)+\left|\Psi_{j}\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\right)T\left(Z,Z'\right)dZ'\right)F'\left[J,\omega,\theta,Z,\Psi\right]}\end{aligned}$$

The successive derivatives (355) in (354) are similar to (319) along with (317) and (318):

$$\left(\frac{\delta^{n}\omega\left(J,\theta,Z\right)}{\prod\limits_{i=1}^{n}\delta\left|\Psi\left(\theta-l_{i},Z_{i}\right)\right|^{2}}\right)_{|\Psi|^{2}=0}=\left(\sum_{m=1}^{n}\sum_{i=1}^{m}\sum_{(line_{1},\ldots,line_{m})}\prod_{i}F\left(line_{i}\right)\prod_{B}F\left(B\right)\right) \tag{356}$$

where $F(line_i)$ is $n\times n$, in other words a $(1,1)$ tensor, given by:

$$\left[F\left(line_{i}\right)\right]_{ab}=\left[\prod_{l=1}^{L_{i}}\hat{T}\left(\theta-\sum_{j=1}^{l-1}\frac{\left|Z^{(j-1)}-Z^{(j)}\right|}{c},Z^{(l-1)},Z^{(l)},\omega_{0},\Psi\right)\right]_{ab}\frac{\omega_{0b}\left(J,\theta-\sum_{l=1}^{L_{i}}\frac{\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z_{i}\right)}{\bar{\mathcal{G}}_{0}\left(0,Z_{i}\right)} \tag{357}$$

To each branching point $(X,\theta)=B$ of valence $k+2$ arising in the expansion, we associate the $(1,k+1)$ tensor:

$$[F((X,\theta))]_{abc_1,\ldots,c_k}=\frac{\delta^k\left(\frac{\frac{\kappa}{N}Tab\left(Z,Z^{(l)}\right)F'\left[J,\theta,\omega_0,Z^{(l)}\right]\bar{\mathcal{G}}_0\left(0,Z^{(l)}\right)}{\omega_{0a}\left(J,\theta,Z^{(l)}\right)}\right)}{\delta\omega_{0c_1}\left(J,\theta,Z^{(l)}\right)\ldots\delta\omega_{0c_k}\left(J,\theta,Z^{(l)}\right)} \tag{358}$$

We attach 1 line, coming in, and $k+1$ lines, coming out, to each branching point. As a consequence, the contraction of a branching point of valence $k+2$ and $k+2$ lines yields a $(1,k+1)$ tensor. The factor associated to the sum of single lines (325) crossing the points Z_k generalizes straightforwardly and is given by:

$$\begin{aligned}
&\hat{T}\left(1-\hat{T}\right)^{-1}\prod_{l=1}^{n-1}\left\{\left(D\left(\left|\Psi\left(\theta-l_l,Z_l\right)\right|^2\right)dZ_l dl_l\right)\hat{T}\left(1-\hat{T}\right)^{-1}\right\}D\left(\left|\Psi\left(\theta-l_n,Z_n\right)\right|^2\omega_0\left(J,\theta-l_n,Z_n\right)\right)\\
=\;&\hat{T}\left(1-\hat{T}\right)^{-1}\frac{1}{1-D\left(\left|\Psi\left(\theta,Z\right)\right|^2\right)\hat{T}\left(1-\hat{T}\right)^{-1}}D\left(\left|\Psi\left(\theta-l_n,Z_n\right)\right|^2\omega_0\left(J,\theta-l_n,Z_n\right)\right)\\
=\;&\hat{T}\frac{1}{1-\left(1+D\left(\left|\Psi\right|^2\right)\right)\hat{T}}D\left(\left|\Psi\left(\theta-l_n,Z_n\right)\right|^2\omega_0\left(J,\theta-l_n,Z_n\right)\right)
\end{aligned} \tag{359}$$

6.3.2 Path integral form for the frequencies

Then, as in section 6.2, the series expansion (356) can be reordered to compute $\omega(J,\theta,Z)$ as a path integral for the action of an auxiliary field $\left(\Lambda,\Lambda^{\dagger}\right)$ with n components. The result is the same as in section 6.2. The action $S(\Lambda)$ is:

$$\begin{aligned}
S(\Lambda)\;=\;&\int\Lambda(Z,\theta)\left(1-D\left(\left|\Psi\right|^2\right)\hat{T}\right)\Lambda^{\dagger}(Z,\theta)\,d(Z,\theta)\\
&-\int\Lambda(Z,\theta)\hat{T}\left(\theta-\frac{\left|Z^{(1)}-Z\right|}{c},Z^{(1)},Z,\omega_0+\hat{T}\Lambda^{\dagger}\right)\Lambda^{\dagger}\left(Z^{(1)},\theta-\frac{\left|Z-Z^{(1)}\right|}{c}\right)dZdZ^{(1)}d\theta^{(1)}
\end{aligned}$$

where $\Lambda(Z,\theta)$ is a two components vector, and $\Lambda^{\dagger}(Z,\theta)$ is the hermitian conjugate. The frequency vector is thus given by the integral:

$$\omega(J,\theta,Z)=\omega_0(J,\theta,Z)+\frac{\int\hat{T}\Lambda^{\dagger}(Z,\theta)\exp\left(-S(\Lambda)+\int\Lambda(X,\theta)D\left(\left|\Psi\right|^2\omega_0(J,\theta,Z)\right)d(X,\theta)\right)\mathcal{D}\Lambda}{\exp\left(-S(\Lambda)\right)\mathcal{D}\Lambda} \tag{360}$$

6.3.3 Saddle path approximation

The solution of (360) is obtained in first approximation by considering that $\Lambda^{\dagger}$ satisfies the saddle point approximation:

$$\left(\left(1-D\left(\left|\Psi\right|^2\right)\hat{T}\right)\Lambda^{\dagger}\right)(Z,\theta)-\left(\hat{T}_{\omega_0+\hat{T}\left(\omega_0\left|\Psi\right|^2\right)}\Lambda^{\dagger}\right)(Z,\theta)-D\left(\left|\Psi\right|^2\omega_0\right)=0 \tag{361}$$

In the sequel, we will define the vector:

$$\frac{\omega(J,\theta,Z)}{\omega(J,\theta,Z)+\hat{T}\left(\left(\left|\Psi\right|^2\omega_0\right)\right)}$$

as the vector with components:

$$\frac{\omega_i(J,\theta,Z)}{\omega_i(J,\theta,Z)+\left(\hat{T}\left(\left(\left|\Psi\right|^2\omega_0\right)\right)\right)_i}$$

More generally, we will define for any F, the vector $VF(\omega,\Psi,...)$ with components $F(\omega_i,\Psi_i,...)$.

Using that:

$$\hat{T}_{\omega_0+\hat{T}\left(\omega_0|\Psi|^2\right)}\simeq D\left(\frac{\omega(J,\theta,Z)}{\omega(J,\theta,Z)+\hat{T}\left(\left(|\Psi|^2\omega_0\right)\right)}\right)\hat{T}_{\omega_0}$$

the saddle point equation (361) becomes:

$$\left(\left(1-D\left(|\Psi|^2\right)\hat{T}\right)\Lambda^\dagger\right)(Z,\theta)-D\left(\frac{\omega_0}{\omega_0+\hat{T}\Lambda^\dagger}\right)\hat{T}\Lambda^\dagger(Z,\theta)-D(\omega_0)|\Psi|^2\simeq 0 \tag{362}$$

Equation (362) can be used in two different ways: first by writing a non-local equation for $\omega(J,\theta,Z)$ and second by solving recursively (362) for an externaly-shaped background field.

6.3.4 Non local equation for $\omega(J,\theta,Z)$

As for the basic case, (362) can be rewritten as an equation for ω. Actually, under the saddle point approximation:

$$\omega(J,\theta,Z)=\omega_0(J,\theta,Z)+\hat{T}\Lambda^\dagger(Z,\theta)$$

and:

$$\hat{T}\Lambda^\dagger(Z,\theta)=\omega(J,\theta,Z)-\omega_0(J,\theta,Z)\equiv\Omega(J,\theta,Z)$$

so that (327) writes:

$$\Lambda^\dagger-(\Omega+\omega_0)|\Psi|^2-\frac{\omega_0}{\omega_0+\Omega}\Omega=0$$

Applying the operator $\hat{T}$ on the left leads to:

$$\Omega-\hat{T}(\Omega+\omega_0)|\Psi|^2-\hat{T}\left(\frac{\omega_0}{\omega_0+\Omega}\right)\Omega=0 \tag{363}$$

where $(\Omega+\omega_0)|\Psi|^2$ and $\left(\frac{\omega_0}{\omega_0+\Omega}\right)\Omega$ are defined as the vectors with components $(\Omega+\omega_0)_i|\Psi|_i^2$ and $\left(\frac{\omega_{i0}}{\omega_{i0}+\Omega_i}\right)\Omega_i$ respectively.

Then we can generalize the expression (231) defining the background field with several component $\Psi_i(\theta,Z)$. We assume a stabilization potential U_i for each component, with minimum X_{i0}. Using the notation V defined after equation (361), the expression for the vector background field becomes:

$$\begin{aligned}\Psi(\theta,Z) &= V\left(\frac{\nabla_\theta\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}{U''(X_0)\,\omega^2\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}\Psi_0(\theta,Z)\right)\\ &= V\left(\frac{\nabla_\theta\omega}{U''(X_0)\,\omega^2}\Psi_0(\theta,Z)\right)\\ &= V\left(\frac{X_0}{U''(X_0)(\omega_0+\Omega)^2}\nabla_\theta\Omega\right)\end{aligned}$$

and the vector of squared norms is:

$$|\Psi|^2=V\left(\frac{X_0^2}{(\omega_0+\Omega)^2U''(X_0)}\nabla_\theta\Omega\right)$$

Equation (332) is then replaced by:

$$\Omega-\hat{T}V\left(\frac{\frac{X_0^2}{U''(X_0)}\nabla_\theta\Omega+\omega_0\Omega}{\omega_0+\Omega}\right)=0 \tag{364}$$

6.3.5 Recursive solution of (362)

Alternatively (362) can be solved recursively for a given bachground field. As in the one component field case, we find in first approximation:

$$\begin{aligned}\hat{T}\Lambda^{\dagger} &= A\frac{1}{1-\left(\hat{T}_{\omega_0+A\omega_0|\Psi|^2}-\hat{T}\right)\hat{T}^{-1}A}\omega_0\left|\Psi\right|^2 \\ &\simeq A\frac{1}{1-D\left(\frac{\omega_0}{\omega_0+A\omega_0|\Psi|^2}\right)A}\omega_0\left|\Psi\right|^2\end{aligned} \tag{365}$$

with:

$$\begin{aligned}A &= \frac{\hat{T}}{1-\left(1+D\left(|\Psi|^2\right)\right)\hat{T}}=\frac{1}{\left(1+D\left(|\Psi|^2\right)\right)}\frac{\left(1+D\left(|\Psi|^2\right)\right)\hat{T}}{1-\left(1+D\left(|\Psi|^2\right)\right)\hat{T}} \\ &= \frac{\hat{T}}{1-\hat{T}-D\left(|\Psi|^2\right)\hat{T}} \\ &= \frac{\hat{T}}{1-\hat{T}}\sum_{n\geqslant 0}\left(D\left(|\Psi|^2\right)\frac{\hat{T}}{1-\hat{T}}\right)^n\end{aligned} \tag{366}$$

and the generalization of (344) is obtained by diagonalization of $\hat{T}$.

6.3.5.1 Case $n=2$ To obtain explicit formula, we consider that $n=2$, that is, there are two type of cells. Writing:

$$\left(1+D\left(|\Psi|^2\right)\right)\hat{T}=\begin{pmatrix}\hat{T}_1\left(\left(1+\left(|\Psi_1|^2\right)\right)\omega_{01}\right) & -g\hat{T}_2\left(\left(1+\left(|\Psi_2|^2\right)\right)\omega_{02}\right)\\ -g\hat{T}_1\left(\left(1+\left(|\Psi_1|^2\right)\right)\omega_{01}\right) & \hat{T}_2\left(\left(1+\left(|\Psi_2|^2\right)\right)\omega_{02}\right)\end{pmatrix}$$

and assuming ω_{01} and ω_{02} changing slowly in time, we have:

$$\left(1+D\left(|\Psi|^2\right)\right)\hat{T}=U\hat{T}_DU^{-1}$$

$$\begin{aligned}\hat{T}_D &= \begin{pmatrix}\frac{1}{2}\left(\hat{T}_1+\hat{T}_2-\sqrt{4g^2\hat{T}_1\hat{T}_2+\left(\hat{T}_1-\hat{T}_2\right)^2}\right) & 0\\ 0 & \frac{1}{2}\left(\hat{T}_1+\hat{T}_2+\sqrt{4g^2\hat{T}_1\hat{T}_2+\left(\hat{T}_1-\hat{T}_2\right)^2}\right)\end{pmatrix} \\ U &= \begin{pmatrix}-\frac{1}{2g}\left(\hat{T}_1-\hat{T}_2-\sqrt{4g^2\hat{T}_1\hat{T}_2+\left(\hat{T}_1-\hat{T}_2\right)^2}\right) & \hat{T}_2\\ \hat{T}_1 & \frac{1}{2g}\left(\hat{T}_1-\hat{T}_2-\sqrt{4g^2\hat{T}_1\hat{T}_2+\left(\hat{T}_1-\hat{T}_2\right)^2}\right)\end{pmatrix}\end{aligned}$$

As a consequence:

$$\hat{T}=UD\left(\frac{\exp\left(-cl_1-\alpha\left(\hat{T}_D\right)\left(\left(cl_1\right)^2-\left|Z-Z_1\right|^2\right)\right)}{B\left(\hat{T}_D\right)}H\left(cl_1-\left|Z-Z_1\right|\right)\right)U^{-1}$$

with $\alpha\left(\hat{T}\right)$ and $B\left(\hat{T}\right)$ are vectors. That is, given our conventions:

$$\hat{T}=U\left(\begin{array}{cc}\frac{\exp\left(-cl_1-\alpha_1\left(\hat{T}_D\right)\left((cl_1)^2-|Z-Z_1|^2\right)\right)}{B_1\left(\hat{T}\right)} & 0\\ 0 & \frac{\exp\left(-cl_1-\alpha_2\left(\hat{T}_D\right)\left((cl_1)^2-|Z-Z_1|^2\right)\right)}{B_2\left(\hat{T}\right)}\end{array}\right)U^{-1}H\left(cl_1-|Z-Z_1|\right)$$

For transfers functions $T_i(Z,Z_1)$ that are proportional $T_i(Z,Z_1)=C_iT_0(Z,Z_1)$, the change of basis yields the diagonalized transfer function:

$$T_D(Z,Z_1)=\left(\begin{array}{cc}\frac{1}{2}\left(C_1+C_2-\sqrt{4g_1^2C_1C_2+(C_1-C_2)^2}\right) & 0\\ 0 & \frac{1}{2}\left(C_1+C_2+\sqrt{4g_1^2C_1C_2+(C_1-C_2)^2}\right)\end{array}\right)T_0(Z,Z_1)$$

Appendix 5.2 shows that $\alpha_i\left(\hat{T}\right)$ and $B_i\left(\hat{T}\right)$ are proportional to the averages of $\hat{T}_{iD}$ and $1+\hat{T}_{iD}$, more precisely:

$$\begin{aligned}D\left(\alpha\left(\hat{T}\right)\right) &\propto \left(\begin{array}{cc}\frac{1}{2}\left(C_1+C_2-\sqrt{4g_1^2C_1C_2+(C_1-C_2)^2}\right) & 0\\ 0 & \frac{1}{2}\left(C_1+C_2+\sqrt{4g_1^2C_1C_2+(C_1-C_2)^2}\right)\end{array}\right)\\ D\left(B\left(\hat{T}\right)\right) &\propto \left(\begin{array}{cc}\frac{1}{2}\left(C_1+C_2-\sqrt{4g_1^2C_1C_2+(C_1-C_2)^2}\right) & 0\\ 0 & \frac{1}{2}\left(C_1+C_2+\sqrt{4g_1^2C_1C_2+(C_1-C_2)^2}\right)\end{array}\right)\end{aligned}$$

As a consequence, by multiplication with U and U^{-1}, we find that:

$$\frac{\left(1+D\left(|\Psi|^2\right)\right)\hat{T}}{1-\left(1+D\left(|\Psi|^2\right)\right)\hat{T}}=\frac{\exp\left(-cl_1-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_1)^2-|Z-Z_1|^2\right)\right)}{B}H\left(cl_1-|Z-Z_1|\right) \tag{367}$$

with:

$$\begin{aligned}\Lambda &= \left(\begin{array}{cc}C_1 & -gC_2\\ -gC_1 & C_2\end{array}\right)\\ B &= 1+2\pi\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\end{aligned}$$

where the constants C_1 and C_2 are as in Appendix 4.3 to define $\hat{T}_1$ and $\hat{T}_2$.

6.3.5.2 General case The formula of the previous paragraph generalize to a system with n interacting components, and with have the generalization of (367):

$$\begin{aligned}A\left((Z,\theta),(Z_1,\theta-l_1)\right) &\simeq D\left(\frac{1}{\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)}\right) \\ &\times\left(\frac{\exp\left(-cl_1-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_1)^2-|Z-Z_1|^2\right)\right)}{B}H\left(cl_1-|Z-Z_1|\right)\right)\end{aligned} \tag{368}$$

As a consequence, the expansion of (365) is:

$$\begin{aligned}\omega(Z,\theta) &= \omega_0(J,\theta,Z)+\int\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}\frac{\exp\left(-cl_i-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_i)^2-\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{B}\\ &\times D\left(\frac{\omega_0(\theta-l_i,Z_i)}{\omega_0(\theta-l_i,Z_i)+A\omega_0|\Psi|^2(\theta-l_i,Z_i)}\frac{\omega_0(J,\theta-l_k,Z_k)}{\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)}\right)\\ &\times\frac{\exp\left(-cl_k-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_i)^2-\frac{|Z_{k-1}-Z_k|}{c}\right)\right)}{B}|\Psi(\theta-l_k,Z_k)|^2dZ_idl_i \quad (369)\end{aligned}$$

$$\begin{aligned}\omega^{-1}(Z,\theta) &= \omega_0^{-1}(J,\theta,Z)\\ &+D\left(\frac{G'[J,\omega,\theta,Z,\Psi]}{F'[J,\omega,\theta,Z,\Psi]}\right)\int\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}\frac{\exp\left(-cl_i-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_i)^2-\frac{|Z_i-Z_{i+1}|}{c}\right)\right)}{D}\\ &\times D\left(\frac{\omega_0(\theta-l_i,Z_i)}{\omega_0(\theta-l_i,Z_i)+A\omega_0|\Psi|^2(\theta-l_i,Z_i)}\frac{\omega_0(J,\theta-l_k,Z_k)}{\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)}\right)\\ &\times\frac{\exp\left(-cl_k-\left(1+D\left(\left\langle|\Psi|^2\right\rangle\right)\right)\Lambda\left((cl_i)^2-\frac{|Z_{k-1}-Z_k|}{c}\right)\right)}{B}|\Psi(\theta-l_k,Z_k)|^2dZ_idl_i \quad (370)\end{aligned}$$

and the full series expansion is obtained as in the previous paragraph:

$$\begin{aligned}&\hat{T}\Lambda^{\dagger}=\sum_{n,p}\sum_{\sum_{i=1}^{p}r_i=n-1}\sum_{l_1,...,l_n,\sum_{i=1}^{n}l_i=p}\sum_{\substack{f\in\{1,...,p\}^{\{1,...,n\}},\\ f(k)\notin\left\{\sum_{i=1}^{k-1}l_i+1,...\sum_{i=1}^{k}l_i\right\}}}A^{l_1}\left(X,\hat{X}_1,...,\hat{X}_{l_1},X_0,X_1'\right)\omega_0|\Psi|^2(X_1')dX_1'\\ &\times\prod_{k=1}^{p}\frac{d\hat{X}_k}{\sharp k!}\times\prod_{k=2}^{n}A^{l_k}\left(X_k,\hat{X}_{\sum_{i=1}^{k-1}l_i+1},...,\hat{X}_{\sum_{i=1}^{k}l_i},X_k'\right)\omega_0|\Psi|^2(X_k')dX_k'\\ &\times\prod_{i}^{p}\frac{\frac{\delta^{r_i}\hat{T}}{\delta^{r_i}\omega_0(\hat{X}_i)}}{\hat{T}}\prod_{k=2}^{n}\delta\left(X_k-\hat{X}_{f(k)}\right) \quad (371)\end{aligned}$$

The corrections due to the fluctuations around the saddle point can be derived as in the previous paragraph, but the computations will be omitted here.

Appendix 7. Dynamic equations for connectivity functions

7.1 General formula

We adapt the description of ([52]) to our context. The transfer function T from i to j satisfies the following equation:

$$\begin{aligned}&\nabla_{\theta^{(i)}(n_i)}T\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) \quad (372)\\ =\;&-\frac{1}{\tau}T\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\\ &+\lambda\left(\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\right)\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z_i-Z_j|}{c}\right)\end{aligned}$$

where $\hat{T}$ measures the variation of T due to the signals send from j to i and the signals emitted by i. It satisfies the following equation:

$$\begin{aligned}
&\nabla_{\theta^{(i)}(n_i)}\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) \qquad (373)\\
=\; &\rho\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z_i-Z_j|}{c}\right)\\
&\times\left\{\left(h(Z,Z_1)-\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\right)C\left(\theta^{(i)}(n-1)\right)h_C(\omega_i(n_i))\right.\\
&\left.-D\left(\theta^{(i)}(n-1)\right)\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)h_D(\omega_j(n_j))\right\}
\end{aligned}$$

where h_C and h_D are increasing functions. We depart slightly from ([52]) by the introduction of the function $h(Z,Z_1)$ (they chose $h(Z,Z_1)=1$), to implement some loss due to the distance. We may chose for example:

$$h(Z,Z_1)=\exp\left(-\frac{|Z_i-Z_j|}{\nu c}\right)$$

where ν is a parameter. Equation (373) involves two dynamic factors $C\left(\theta^{(i)}(n-1)\right)$ and $D(\theta_i(n-1))$. The factor $C\left(\theta^{(i)}(n-1)\right)$ describes the accumulation of input spikes. It is solution of the differential equation:

$$\begin{aligned}
\nabla_{\theta^{(i)}(n-1)}C\left(\theta^{(i)}(n-1)\right) &= -\frac{C\left(\theta^{(i)}(n-1)\right)}{\tau_C} \qquad (374)\\
&+\alpha_C\left(1-C\left(\theta^{(i)}(n-1)\right)\right)\omega_j\left(Z_j,\theta^{(i)}(n-1)-\frac{|Z_i-Z_j|}{c}\right)
\end{aligned}$$

In the continuous approximation, the solution of (374) is:

$$\begin{aligned}
C\left(\theta^{(i)}(n-1)\right) &= \int\exp\left(-\left(\frac{\left(\theta^{(i)}(n-1)-\theta^{(i)\prime}\right)}{\tau_C}+\alpha_C\int_{\theta^{(i)\prime}}^{\theta^{(i)}(n-1)}\omega_j\left(Z_j,\theta'-\frac{|Z_i-Z_j|}{c}\right)d\theta'\right)\right)\\
&\times\omega_j\left(Z_j,\theta^{(i)\prime}-\frac{|Z_i-Z_j|}{c}\right)d\theta^{(i)\prime}
\end{aligned}$$

If a static equilibrium $\omega_0(Z_j)$ exists, expanding around this equilibrium leads to approximate the integral:

$$\int_{\theta_i'}^{\theta_i}\omega_j\left(Z_j,\theta'-\frac{|Z_i-Z_j|}{c}\right)d\theta'$$

by the quantity:

$$\omega_0(Z_j)\left(\theta^{(i)}(n-1)-\theta^{(i)\prime}\right)$$

so that:

$$\begin{aligned}
C\left(\theta^{(i)}(n-1)\right) &= \int\exp\left(-\left(\frac{1}{\tau_C}+\alpha_C\omega_0(Z_j)\right)\left(\theta^{(i)}(n-1)-\theta^{(i)\prime}\right)\right) \qquad (375)\\
&\times\left(C_0+\omega_j\left(Z_j,\theta^{(i)\prime}-\frac{|Z_i-Z_j|}{c}\right)\right)d\theta_i'
\end{aligned}$$

The term $D(\theta_i(n-1))$ is proportional to the accumulation of output spikes and is solution of:

$$\nabla_{\theta^{(i)}(n-1)}D\left(\theta^{(i)}(n-1)\right)=-\frac{D\left(\theta^{(i)}(n-1)\right)}{\tau_D}+\alpha_D\left(1-D\left(\theta^{(i)}(n-1)\right)\right)\omega_i(Z_i) \qquad (376)$$

In the continuous approximation, the solution of (376) is:

$$D\left(\theta^{(i)}(n-1)\right)=\int\exp\left(-\left(\frac{1}{\tau_D}+\alpha_D\omega_0(Z_i)\right)\left(\theta^{(i)}(n-1)-\theta^{(i)\prime}\right)\right)\left(D_0+\omega_i\left(Z_i,\theta^{(i)\prime}\right)\right)d\theta^{(i)\prime} \qquad (377)$$

As a consequence, the dynamics for transfer functions is a set of two equations:

$$\begin{aligned}
&\nabla_{\theta^{(i)}(n_i)} T\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) \\
=\;& -\frac{1}{\tau}T\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) \\
&+\lambda\left(\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\right)\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z_i-Z_j|}{c}\right)
\end{aligned} \tag{378}$$

and:

$$\begin{aligned}
&\nabla_{\theta^{(i)}(n_i)} \hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) \\
=\;& \rho\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z_i-Z_j|}{c}\right) \\
&\times\left\{\left(h(Z,Z_1)-\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\right)C\left(\theta^{(i)}(n-1)\right)h_C(\omega_i(n_i))\right. \\
&\left.-D\left(\theta^{(i)}(n-1)\right)\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)h_D(\omega_j(n_j))\right\}
\end{aligned} \tag{379}$$

with $C\left(\theta^{(i)}(n-1)\right)$ and $D\left(\theta^{(i)}(n-1)\right)$ given by (375) and (377).

The field translation of (378) and (379) is obtained by including the following potential terms in the action for the field:

$$\begin{aligned}
&\int\left(\nabla_\theta T((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))+\frac{T((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))}{\tau}\right. \\
&\left.-\lambda\left(\hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))\right)\delta\left(\theta-\theta_1-\frac{|Z-Z_1|}{c}\right)\right) \\
&\times|\Psi(\theta,Z,\omega)|^2|\Psi(\theta_1,Z_1,\omega_1)|^2
\end{aligned} \tag{380}$$

corresponding to (378) and:

$$\begin{aligned}
&\int\left(\nabla_\theta\hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))-\rho\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z-Z_1|}{c}\right)\right. \\
&\left.\times\left\{\left(h(Z,Z_1)-\hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))\right)C(\theta,Z,Z_1)h_C(\omega)-D(\theta,Z)\hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))h_D(\omega_1)\right\}\right) \\
&\times|\Psi(\theta,Z,\omega)|^2|\Psi(\theta_1,Z_1,\omega_1)|^2
\end{aligned} \tag{381}$$

for (379), with $C(\theta,Z,Z_1)$ and $D(\theta,Z)$ are defined as:

$$C(\theta,Z,Z_1)=\int^\theta\exp\left(-\left(\frac{1}{\tau_C}+\alpha_C\omega_0(Z_1)\right)(\theta-\theta')\right)\left(C_0+\omega\left(Z_1,\theta'-\frac{|Z-Z_1|}{c}\right)\right)d\theta'$$

$$D(\theta,Z)=\int^\theta\exp\left(-\left(\frac{1}{\tau_D}+\alpha_D\omega_0(Z)\right)(\theta-\theta')\right)(D_0+\omega(Z,\theta'))d\theta'$$

For

$$\begin{aligned}
\tau_C(Z_1) &= \frac{1}{\tau_C}+\alpha_C\omega_0(Z_1)<1 \\
\tau_D(Z) &= \frac{1}{\tau_D}+\alpha_D\omega_0(Z)<1
\end{aligned}$$

and if the transfer function adapts slowly with respect to $\omega(Z,\theta)$, we can simplify the expressions for $C(\theta,Z,Z_1)$ and $D(\theta,Z)$:

$$\begin{aligned} C(\theta, Z, Z_1) &\simeq C(Z_1) = \frac{C_0 + \omega_0(Z_1)}{\tau_C(Z_1)} \\ D(\theta, Z) &\simeq D(Z) = \frac{D_0 + \omega_0(Z)}{\tau_D(Z)} \end{aligned}$$

After projection on the dependent frequency states the transfer functions become functions $T((Z,\theta),(Z_1,\theta_1))$ and $\hat{T}((Z,\theta),(Z_1,\theta_1))$ respectively. Moreover, we can simplify the action by finding the configurations for $T((Z,\theta),(Z_1,\theta_1))$ and $\hat{T}((Z,\theta),(Z_1,\theta_1))$ that minimize the potential terms (380), (381). It corresponds to set:

$$\begin{aligned} 0 &= \nabla_\theta T((Z,\theta,\omega),(Z_1,\theta_1,\omega_1)) + \frac{T((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))}{\tau} \\ &\quad -\lambda\left(\hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))\right)\delta\left(\theta - \theta_1 - \frac{|Z - Z_1|}{c}\right) \end{aligned} \tag{382}$$

and:

$$\begin{aligned} 0 &= \left(\nabla_\theta \hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1)) - \rho\delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z - Z_1|}{c}\right)\right. \\ &\quad \times\left\{\left(h(Z,Z_1) - \hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))\right)C(\theta,Z,Z_1)h_C(\omega) - D(\theta,Z)\hat{T}((Z,\theta,\omega),(Z_1,\theta_1,\omega_1))h_D(\omega_1)\right\} \end{aligned} \tag{383}$$

We look for solutions of the form:

$$\begin{aligned} T\left(Z,\theta,Z_1,\theta - \frac{|Z - Z_1|}{c}\right) &\equiv T(Z,\theta,Z_1) \\ \hat{T}\left(Z,\theta,Z_1,\theta - \frac{|Z - Z_1|}{c}\right) &\equiv \hat{T}(Z,\theta,Z_1) \end{aligned}$$

so that $T(Z,\theta,Z_1)$ and $\hat{T}(Z,\theta,Z_1)$ satisfy:

$$\nabla_\theta T(Z,\theta,Z_1) + \left(\frac{T(Z,\theta,Z_1)}{\tau} - \lambda\hat{T}(Z,\theta,Z_1)\right) = 0 \tag{384}$$

$$\begin{aligned} &\nabla_\theta \hat{T}(Z,\theta,Z_1) \\ &= \rho\left(\left(h(Z,Z_1) - \hat{T}(Z,\theta,Z_1)\right)C(Z_1)h_C(\omega(Z,\theta)) - \hat{T}(Z,\theta,Z_1)D(Z)h_D\left(\omega\left(Z_1,\theta - \frac{|Z - Z_1|}{c}\right)\right)\right) \end{aligned} \tag{385}$$

Using (384), we replace $\hat{T}(Z,\theta,Z_1)$ in (385):

$$\hat{T}(Z,\theta,Z_1) = \frac{\nabla_\theta T(Z,\theta,Z_1)}{\lambda} + \frac{T(Z,\theta,Z_1)}{\lambda\tau}$$

and we arrive to the differential equation satisfied by $T(Z,\theta,Z_1)$:

$$\frac{\nabla_\theta^2 T(Z,\theta,Z_1)}{\lambda} + U_1(\omega)\nabla_\theta T(Z,\theta,Z_1) + U_2(\omega)T(Z,\theta,Z_1) = \rho C(Z_1)h(Z,Z_1)h_C(\omega(Z,\theta)) \tag{386}$$

where:

$$\begin{aligned} U_1(\omega) &= \left(\frac{1}{\lambda\tau} + \frac{\rho}{\lambda}\left(C(Z_1)h_C(\omega(Z,\theta)) + D(Z)h_D\left(\omega\left(Z_1,\theta - \frac{|Z - Z_1|}{c}\right)\right)\right)\right) \\ U_2(\omega) &= \frac{\rho}{\lambda\tau}\left(C(Z_1)h_C(\omega(Z,\theta)) + D(Z)h_D\left(\omega\left(Z_1,\theta - \frac{|Z - Z_1|}{c}\right)\right)\right) \end{aligned}$$

If we consider that the transfer function varies slowly compared to the oscillations of the thread, we can approximate (386) by a quite static equation:

$$U_2(\omega)\,T(Z,\theta,Z_1)=\rho C(Z_1)\,h(Z,Z_1)\,h_C(\omega(Z,\theta))$$

whose solution is:

$$\begin{aligned}T(Z,\theta,Z_1) &= \frac{\lambda\tau C(Z_1)\,h(Z,Z_1)\,h_C(\omega(Z,\theta))}{C(Z_1)\,h_C(\omega(Z,\theta))+D(Z)\,h_D\left(\omega\left(Z_1,\theta-\frac{|Z-Z_1|}{c}\right)\right)} \\ &\simeq \frac{\lambda\tau h(Z,Z_1)}{1+\frac{D(Z)}{C(Z_1)}\frac{h_D\left(\omega\left(Z_1,\theta-\frac{|Z-Z_1|}{c}\right)\right)}{h_C(\omega(Z,\theta))}}\end{aligned} \tag{387}$$

Thus $T(Z,\theta,Z_1)$ is a decreasing function of $\omega\left(Z_1,\theta-\frac{|Z-Z_1|}{c}\right)$ and an increasing function of $\omega(Z,\theta)$, as hypothesized in the text. The fully static solution associated to (387) is:

$$T_0(Z,Z_1)=\frac{\lambda\tau h(Z,Z_1)}{1+\frac{D(Z)}{C(Z_1)}\frac{h_D(\omega_0(Z_1))}{h_C(\omega_0(Z))}}$$

7.2 Linearized dynamics

We conclude this section by giving the linearized version of (386) around the static solution $(\omega_0(Z),T_0(Z,Z_1))$. It is:

$$\begin{aligned}0 &= \frac{\nabla_\theta^2 T(Z,\theta,Z_1)}{\lambda}+U_1(\omega_0)\,\nabla_\theta T(Z,\theta,Z_1)+U_2(\omega_0)\,T(Z,\theta,Z_1) \\ &\quad -\rho C(Z_1)\left(1-\frac{T_0(Z,Z_1)}{\lambda\tau}\right)h_C'(\omega_0(Z))\,\Omega(Z,\theta) \\ &\quad +\frac{\rho T_0(Z,Z_1)}{\lambda\tau}\left(D(Z)\,h_D'(\omega_0(Z_1))\,\Omega\left(Z_1,\theta-\frac{|Z-Z_1|}{c}\right)\right)\end{aligned} \tag{388}$$

where:

$$\begin{aligned}U_1(\omega_0) &= \frac{1}{\lambda\tau}+\frac{\rho}{\lambda}\left(C(Z_1)\,h_C(\omega_0(Z))+D(Z)\,h_D(\omega_0(Z_1))\right) \\ U_2(\omega_0) &= \frac{\rho}{\lambda\tau}\left(C(Z_1)\,h_C(\omega_0(Z))+D(Z)\,h_D(\omega_0(Z_1))\right) \\ T_0(Z,Z_1) &= \frac{\lambda\tau h(Z,Z_1)}{1+\frac{D(Z)}{C(Z_1)}\frac{h_D(\omega_0(Z_1))}{h_C(\omega_0(Z))}} \\ T_0(Z,Z_1) &= \frac{T_0(Z,Z_1)}{h(Z,Z_1)} \\ T(Z,\theta,Z_1) &= \frac{T(Z,\theta,Z_1)-T_0(Z,Z_1)}{h(Z,Z_1)} \\ \Omega(Z,\theta) &= \omega(Z,\theta)-\omega_0(Z)\end{aligned}$$

for $\omega_0(Z)\equiv\omega_0$, this reduces to:

$$\begin{aligned}&\frac{\nabla_\theta^2 T(Z,\theta,Z_1)}{\lambda}+U_1(\omega_0)\,\nabla_\theta T(Z,\theta,Z_1)+U_2(\omega)\,T(Z,\theta,Z_1) \\ = &-\frac{\rho T_0(Z,Z_1)\,D(Z)\,h_D'(\omega_0)\,\Omega\left(Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\lambda\tau}+\rho C(Z_1)\left(1-\frac{T_0(Z,Z_1)}{\lambda\tau}\right)h_C'(\omega_0)\,\Omega(Z,\theta)\end{aligned} \tag{389}$$

where:

$$\begin{aligned}
U_1(\omega) &= \frac{1}{\lambda\tau}+\frac{\rho}{\lambda}\left(Ch_C(\omega_0)+Dh_D(\omega_0)\right)\\
U_2(\omega) &= \frac{\rho}{\lambda\tau}\left(Ch_C(\omega_0)+Dh_D(\omega_0)\right)\\
T_0(Z,Z_1) &= \frac{\lambda\tau h(Z,Z_1)}{1+\frac{D}{C}\frac{h_D(\omega_0)}{h_C(\omega_0)}}\\
C &= \frac{C_0+\omega_0}{\tau_C}\\
D &= \frac{D_0+\omega_0}{\tau_D}
\end{aligned}$$

which can also be written, up to the second order in derivatives:

$$\begin{aligned}
&\frac{\nabla_\theta^2 T(Z,\theta,Z_1)}{\lambda}+U_1(\omega_0)\nabla_\theta T(Z,\theta,Z_1)+U_2(\omega)T(Z,\theta,Z_1) \qquad (390)\\
=&\left(\rho C(Z_1)h'_C(\omega_0)-\frac{\rho T_0(Z,Z_1)\left(D(Z)h'_D(\omega_0)+C(Z_1)h'_C(\omega_0)\right)}{\lambda\tau}\right)\Omega(Z,\theta)\\
&+\frac{\rho T_0(Z,Z_1)D(Z)h'_D(\omega_0)\left(\frac{|Z-Z_1|}{c}\nabla_\theta\Omega(Z,\theta)-\frac{(Z-Z_1)^2}{2c^2}\nabla_\theta^2\Omega(Z,\theta)-\frac{(Z-Z_1)^2}{2}\nabla_Z^2\Omega(Z,\theta)\right)}{\lambda\tau}
\end{aligned}$$

Then, to separate the dependences in time and position, we define:

$$\begin{aligned}
T(Z,\theta) &= \int h(Z,Z_1)\frac{T(Z,\theta,Z_1)}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2+\frac{\pi}{2}\alpha}}\\
\bar{C}(Z) &= \frac{1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2+\frac{\pi}{2}\alpha}}\int h(Z,Z_1)C(Z_1)\\
\bar{C}_0(Z) &= \frac{1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2+\frac{\pi}{2}\alpha}}\int h(Z,Z_1)C(Z_1)T_0(Z,Z_1)\\
T_0(Z) &= \frac{1}{\sqrt{\frac{\pi}{8}\left(\frac{1}{X_r}\right)^2+\frac{\pi}{2}\alpha}}\int h(Z,Z_1)T_0(Z,Z_1)
\end{aligned}$$

and $T(Z,\theta)$ satisfies:

$$\begin{aligned}
&\frac{\nabla_\theta^2 T(Z,\theta)}{\lambda}+U_1(\omega_0)\nabla_\theta T(Z,\theta)+U_2(\omega)T(Z,\theta)\\
=&\left(\rho\bar{C}(Z)h'_C(\omega_0)-\frac{\rho\left(D(Z)T_0(Z)h'_D(\omega_0)+\bar{C}_0(Z)h'_C(\omega_0)\right)}{\lambda\tau}\right)\Omega(Z,\theta)\\
&+\frac{\rho D(Z)h'_D(\omega_0)\left(\Gamma_1\nabla_\theta\Omega(Z,\theta)-\left(\Gamma_1\nabla_\theta^2\Omega(Z,\theta)+c^2\Gamma_2\nabla_Z^2\Omega(Z,\theta)\right)\right)}{\lambda\tau}
\end{aligned}$$

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